Quantizing KG field $\phi(\vec{x})$

Start with $\mathcal{H} = \frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2\phi^2}{2}$

$\phi(\vec{x})$ & $\pi(\vec{x})$ – conjugate coord. & momentum (density)

$\phi(\vec{x})$, $\pi(\vec{x})$ – opts $\quad [\phi(\vec{x}), \pi(\vec{x}')] = i\,\delta^{(3)}(\vec{x}-\vec{x}')$

<u>That's it!</u>

Note! no time-dependence so far $\quad \dot{\phi} = \dot{\pi} = 0$

& KG eq. doesn't hold yet $\quad \ddot{\phi} \neq (\nabla^2 - m^2)\phi$

Consequences

FT of $\phi(\vec{x})$ & $\pi(\vec{x})$ is of the form

$$\begin{cases} \phi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{+}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right) \\ \pi(\vec{x}) = \int_{\vec{p}} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^{+}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right) \end{cases}$$

$$\left[a_{\vec{p}}, a^{+}_{\vec{p}'}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

$$H = \int_{\vec{p}} \omega_{\vec{p}}\, a^{+}_{\vec{p}} a_{\vec{p}} + \text{const} \qquad \left[H, a^{+}_{\vec{p}}\right] = \omega_{\vec{p}}\, a^{+}_{\vec{p}} \quad \left[H, a_{\vec{p}}\right] = -\omega_{\vec{p}}\, a_{\vec{p}}$$

$$\vec{P} = \int_{\vec{p}} \vec{p}\, a^{+}_{\vec{p}} a_{\vec{p}} \quad \left[= -\int_{\vec{x}} \pi(\vec{x}) \nabla \phi(\vec{x}) \right]$$

$$H = \int_{\vec{p}} \omega_p \left(a_p^+ a_p + \frac{1}{2} \underbrace{[a_p, a_p^+]}_{\delta(0)} \right)$$

$$\text{const} = \int_{\vec{p}} \frac{\omega_p}{2} \delta(0)$$

$$\vec{P} = \int_{\vec{p}} \vec{p} \left(a_p^+ a_p + \frac{1}{2} \delta(0) \right)$$

$$\text{const} = \underbrace{\int_{\vec{p}} \frac{\vec{p}}{2} \delta(0)}_{\delta(0) \cdot 0} = 0$$

Vacuum $|0\rangle$: $a_p|0\rangle = 0 \;\forall\, \vec{p}$ (Drop $\rightarrow$ in a_p)

$a_p^\dagger$ creates a boson with momentum $\vec{p}$ & energy ω_p

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2} \equiv E_p \qquad (E_p, \vec{p})$$

Normalization : $\langle 0|0\rangle = 1$

$$|\vec{p}\rangle \propto a_p^\dagger |0\rangle$$

$$\langle \vec{p}\,|\,\vec{q}\,\rangle = (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{q}) \rightarrow \text{not Lorentz inv}$$

$$E_p\, \delta^{(3)}(\vec{p} - \vec{q}) - \text{Lorentz inv}$$

$\hookrightarrow$ check

Define $\quad |\vec{p}\rangle = \sqrt{2E_p}\; a_p^\dagger\, |0\rangle$

$$\langle \vec{p} | \vec{q} \rangle = 2E_p\,(2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{q})$$

$$(\mathbb{1})_{\text{1-particle}} = \int_{\vec{p}} |\vec{p}\rangle \frac{1}{2E_p} \langle \vec{p}|$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\, \delta(p^2 - m^2) \Big|_{p^0 > 0}$$

$$p^2 = (p^0)^2 - |\vec{p}|^2 \qquad \delta\Big((p^0)^2 \underbrace{- |\vec{p}|^2 - m^2}_{-E_p^2}\Big)$$

$$\delta\left((p^0)^2 - E_p^2\right) =$$

$$= \delta\left[(p^0 - E_p)(p^0 + E_p)\right] = \frac{1}{2E_p}\, \delta(p^0 - E_p)$$

$$\text{''} \quad \delta\left[2E_p (p^0 - E_p)\right]$$

$$\delta(c\,x) = \frac{1}{|c|}\, \delta(x)$$

$$\phi(\vec{x})|0\rangle = \int_{\vec{p}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\cancel{a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}}} + a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)|0\rangle =$$

$$= \int_{\vec{p}} \frac{1}{2E_p} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$$

$$|\vec{x}\rangle = \int_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$$

$$a^{\dagger}_p |0\rangle = \frac{|\vec{p}\rangle}{\sqrt{2E_p}}$$

$$x\,\psi(x) = x_0\,\psi(x)$$

$$\psi(x) = \delta(\vec{x} - \vec{x}_0)$$

$$E_p = \sqrt{|\vec{p}|^2 + m^2}$$

Schrödinger $\longrightarrow$ Heisenberg

$$\phi(x) = \phi(\vec{x}, t) = e^{iHt}\, \phi(\vec{x})\, e^{-iHt}$$

Similarly $\quad \pi(x) = \pi(\vec{x}, t)$

Eqs. of motion $\qquad i\,\dfrac{\partial O}{\partial t} = [O, H]$

$$i\,\frac{\partial \phi}{\partial t} = \left[\phi(x),\ \int d^3x' \,\frac{\pi^2(\vec{x}', t)}{2}\right]$$

$$[\phi, \pi^2] = [\phi, \pi]\,\pi + \pi\,[\phi, \pi] = 2i\,\delta^{(3)}(\vec{x} - \vec{x}')\,\pi$$

$$\frac{\partial \phi}{\partial t} = \pi = \frac{\delta H}{\delta \pi}$$

$$\frac{\partial \pi}{\partial t} = -\left(-\nabla^2 + m^2\right)\phi = -\frac{\delta H}{\delta \phi}$$

$$\ddot{\phi} = \left(\nabla^2 - m^2\right)\phi \quad \text{KG eq.}$$

$$e^{iHt} a_p e^{-iHt} = a_p e^{i(\cancel{H} - E_p)t} \cancel{e^{-iHt}} = a_p e^{-iE_p t}$$

$$[H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}}$$

$$e^A e^B = e^{A+B}$$

$$\text{if } [A, B] = 0$$

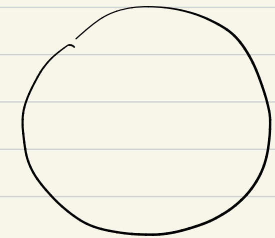
$$H^n a_p = a_p (H - E_p)^n$$

$$H^2 a_p = H a_p (H - E_p) = a_p (H - E_p)^2$$

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}$$

$$p^\mu p_\mu = m^2$$

$$e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}$$



$$\phi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right)$$

positive freq

negative freq

$$-E_p t + \vec{p}\cdot\vec{x} = p\cdot x$$

$$\phi(\vec{x},t) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x} \right)\Big|_{p^0=E_p}$$

$$\pi(\vec{x},t) = \frac{\partial \phi(\vec{x},t)}{\partial t}$$

$$\phi(\vec{x},0) = \phi(\vec{x})$$

$U^{+} \, O \, U$

Heisenberg Schrödinger

$$\phi(\vec{x},t) = e^{-iHt}\,\phi(\vec{x},0)\,e^{iHt}$$

$$e^{-i\vec{P}\cdot\vec{x}}\,\phi(0,t)\,e^{i\vec{P}\cdot\vec{x}} = \phi(\vec{x},t)$$

$$e^{-\vec{P}\cdot\vec{x}}\,a_p\,e^{i\vec{P}\cdot\vec{x}} = a_p\,e^{i\vec{p}\cdot\vec{x}}$$

Total 4-momentum $P^{\mu} = (H, \vec{P})$

$$\phi(x) = e^{iP\cdot x}\,\phi(0)\,e^{-iP\cdot x}$$