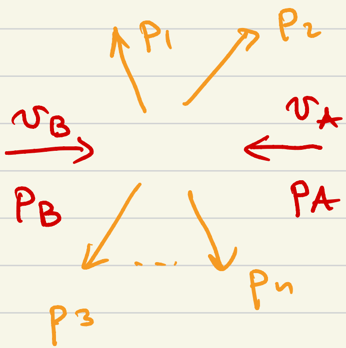
Recall:

$$|in\rangle = |p_A \, p_B\rangle \qquad |out\rangle = |p_1 \cdots p_n\rangle$$



S-matrix:

$$\langle out|\, S \,|in\rangle = \lim_{T\to\infty} \langle out|\, \underbrace{e^{-iH(2T)}}_{= \, U(T,-T)} \,|in\rangle$$

$$U(T,-T) = T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]$$

$$S = \underbrace{\mathbb{1}}_{\text{trivial part, no scattering}} + i\underbrace{T}_{\text{due to int}}$$

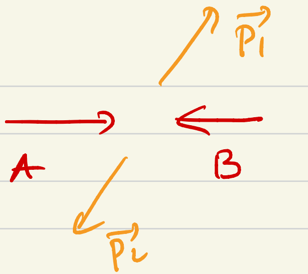
$$\langle out|\, iT \,|in\rangle = (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right) i\mathcal{M}\left(p_A, p_B \to p_f\right)$$

$$\langle \text{out} | iT | \text{in} \rangle = (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right) i\mathcal{M}(p_A, p_B \to p_f)$$

$i\mathcal{M}$ – invariant matrix element, analogous to scattering amplitude of one-particle quantum mechanics.

$$d\sigma = \underbrace{\frac{1}{2E_A 2E_B |v_A - v_B|}}_{\text{transforms as } T^{xy}} \underbrace{\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right)}_{\text{Lorentz-inv. after int. over } p_f}$$

$$\int d\Pi_n = \int \prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right)$$



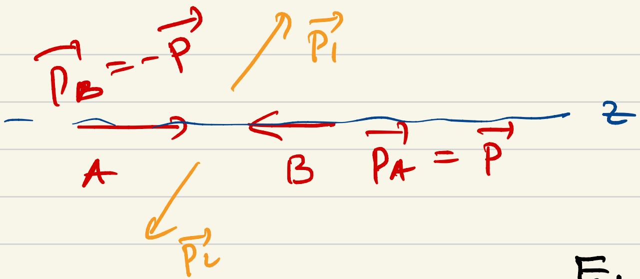
CM: $\vec{p}_1 = -\vec{p}_2$

$$\int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2}\,(2\pi)\,\delta(E_{cm} - E_1 - E_2)$$

$$E_1 = \sqrt{p_1^2 + m_1^2} \qquad E_2 = \sqrt{p_1^2 + m_2^2}$$

$$\int d\Pi_2 = \int d\Omega\, \frac{1}{16\pi^2}\, \frac{|\vec{p}_1|}{E_{cm}}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_A\, 2E_B\, |v_A - v_B|}\, \frac{|p_1|}{(2\pi)^2\, 4\, E_{cm}}\, |\mathcal{M}|^2$$



$$|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_A| = |\vec{p}_B|$$

$$E_A = E_B = \frac{E_{cm}}{2}$$

$$E_A E_B |v_A - v_B| = |E_B p + E_A p| = E_{cm}\, p$$

$$v_A = \frac{p}{E_A}, \qquad v_B = -\frac{p}{E_B}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}$$

Need $\mathcal{M}$ $\longrightarrow$ need $\langle out | S | in \rangle$

$$\lim_{T\to\infty\ldots} \langle p_1 \cdots p_4 | T \, e^{-i\int_{-T}^{T} dt\, H_I(t)} | p_A p_B \rangle$$

From now on ϕ^4 and let $n = 2$

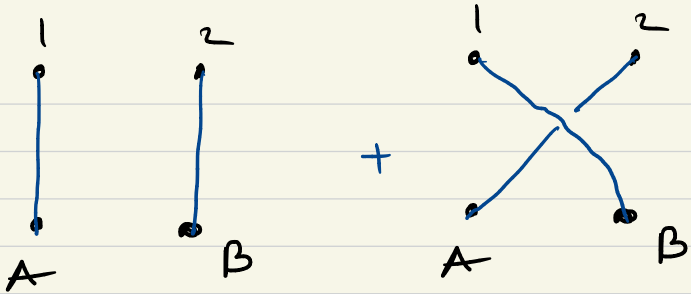
$|p\rangle = 2E_p a_p^\dagger |0\rangle$

0-th order

$$\langle p_1 p_2 | p_A p_B \rangle \propto \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle \propto$$

$$\propto \left[\delta(\vec{p}_1 - \vec{p}_A)\, \delta(\vec{p}_2 - \vec{p}_B) + \delta(\vec{p}_1 - \vec{p}_B)\, \delta(\vec{p}_2 - \vec{p}_A) \right]$$

$|out\rangle = |in\rangle$ - part I


1
2
A
B
+
1
2
A
B

$$\lim_{T\to\infty\ldots} \langle p_1 \cdots p_n | \, T \, e^{-i\int_{-T}^{T} dt\, H_I(t)} \, | p_A p_B \rangle$$

$$\langle p_1 p_2 | \, T \left(-i \frac{\lambda}{4!} \int d^4x \, \phi^4(x) \right) | p_A p_B \rangle =$$

$$= \langle p_1 p_2 | \, N \left(-i \frac{\lambda}{4!} \int d^4x \, \phi^4 + \text{contractions} \right) | p_A p_B \rangle$$

Recall:

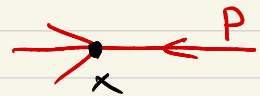
$$\phi(x) = \underset{\substack{\uparrow \\ a_k \text{ only}}}{\phi^+(x)} + \underset{\substack{\uparrow \\ a_k^\dagger \text{ only}}}{\phi^-(x)}$$

$$\langle p | \phi^-$$

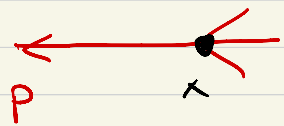
$$\phi^+ |p\rangle = \int_{\vec{k}} \frac{1}{\sqrt{2E_k}} a_k e^{-ikx} \sqrt{2E_p}\, a_p^\dagger |0\rangle = e^{-ip\cdot x} |0\rangle$$

Contraction external states

$$\phi\,|p\rangle = e^{-ip\cdot x} =$$



$$\langle p|\,\phi = e^{ip\cdot x} =$$

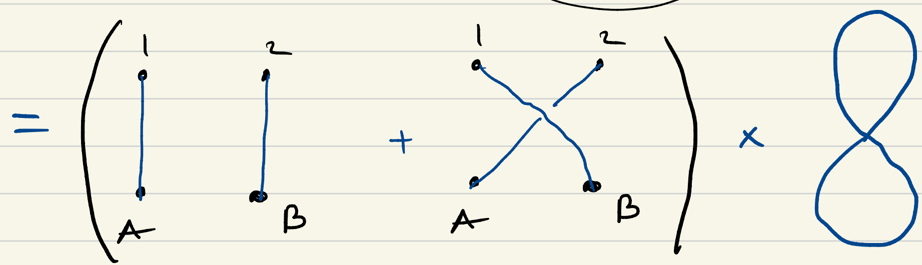


$\phi\,\phi\,\phi\,\phi$ $\qquad\qquad$ $\overbrace{\phi\,\phi}\,\phi\,\phi$ $\qquad\qquad$ $\overbrace{\phi\,\phi}\;\overbrace{\phi\,\phi}$

C $\qquad\qquad\qquad\qquad$ B $\qquad\qquad\qquad\qquad$ A

A :

$$-i\frac{\lambda}{4!}\int d^4x \, \langle p_1 p_2 | \, \overbrace{\phi\,\phi}\;\overbrace{\phi\,\phi} \, | p_A \, p_B \rangle =$$

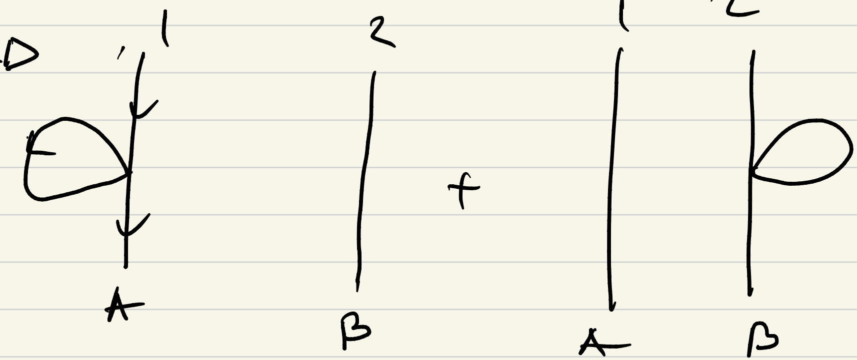


$|out\rangle = |in\rangle$ — part of $\mathbb{1}$

$$B:\ N(\phi\phi\phi\phi) \sim a^+a^+ + \underline{2a^+a} + aa \qquad \langle p_1 | p_A\rangle$$

$$\langle p_1 \underline{p_2}| a^+ \qquad a\,| p_A \underline{p_B}\rangle \to \delta(\vec{p}_1 - \vec{p}_A)$$

$\Rightarrow$



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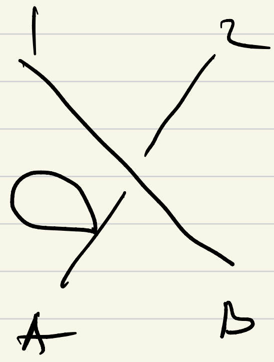

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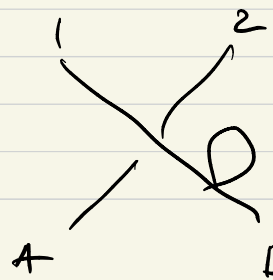
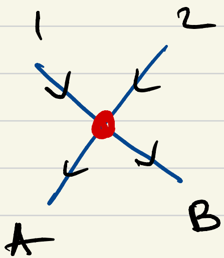


$|00\rangle = |11\rangle$ — part II

C: $\phi\phi\phi\phi$ – contract two ϕ's with $|p_A\rangle\,|p_B\rangle$ and two ϕ's with $|p_1\rangle\,|p_2\rangle$



$$= 4!\left(-\frac{i\lambda}{4!}\right)\int d^4x\; e^{-i(p_A+p_B-p_1-p_2)\cdot x} =$$

$$= -i\lambda\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$\langle \text{out}|\,iT\,|\text{in}\rangle = (2\pi)^4\,\delta^{(4)}\Big(p_A+p_B-\sum p_f\Big)\; i\mathcal{M}(p_A, p_B \to p_f)$$

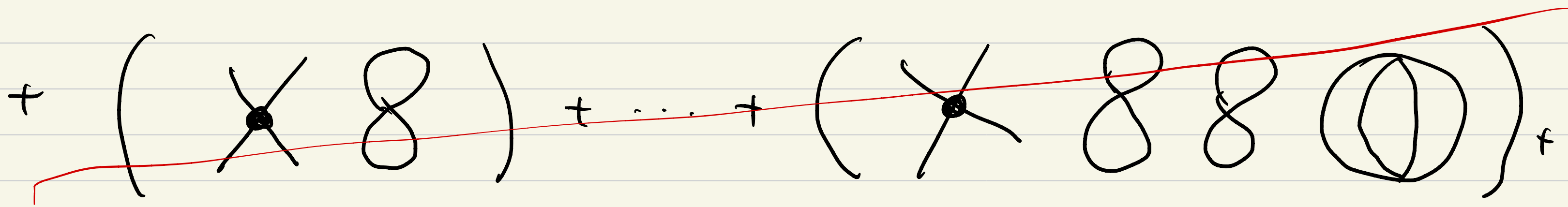
$$\mathcal{M} = -\lambda$$

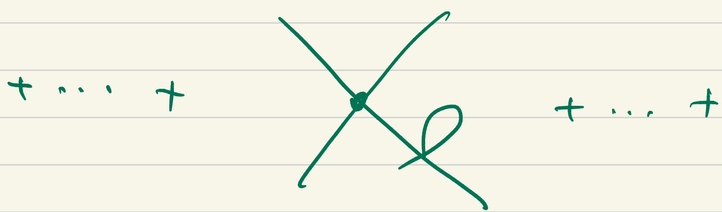
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

$$\sigma_{tot} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

(a)

corrections to (a)





~~diagrams where external lines aren't connected~~

$$\frac{\langle 0| T\phi\phi\, U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} = \langle \Omega | T\phi\phi |\Omega\rangle \qquad |\Omega\rangle \to |0\rangle$$

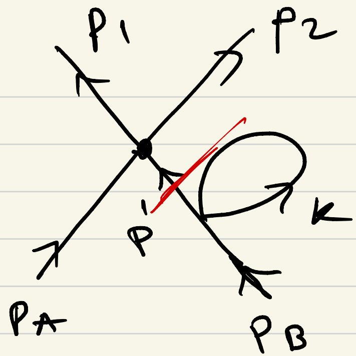
Here $\langle out | S | in\rangle = \langle out | U(T,-T) | in\rangle$

Suppose $|in\rangle = |out\rangle \Rightarrow S = 1$

$$|in\rangle = a_A^\dagger a_B^\dagger |0\rangle \longrightarrow a_A^\dagger a_B^\dagger |\Omega\rangle$$

$|out\rangle = \ldots$

$$\langle out | S | in\rangle = \lim_{T\to\ldots} \frac{\langle 0 | a_1 a_2 U(T,-T) a_A^\dagger a_B^\dagger |0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

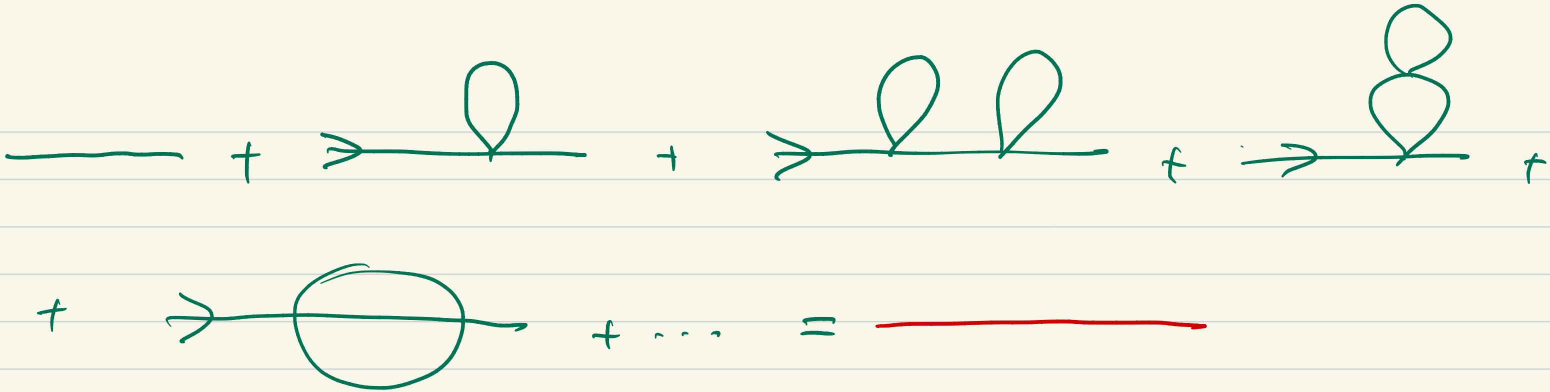


$$= \int \frac{d^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \times$$

$$\times \; (-i\lambda)(2\pi)^4 \delta^{(4)}(p_A + p' - p_1 - p_2)$$

$$\times \; (-i\lambda)(2\pi)^4 \delta^{(4)}(p' - p_B) \qquad \propto$$

$$\propto \quad \frac{1}{p_B^2 - m^2} = \frac{1}{0} \qquad b/c \quad p_B^2 = m^2$$

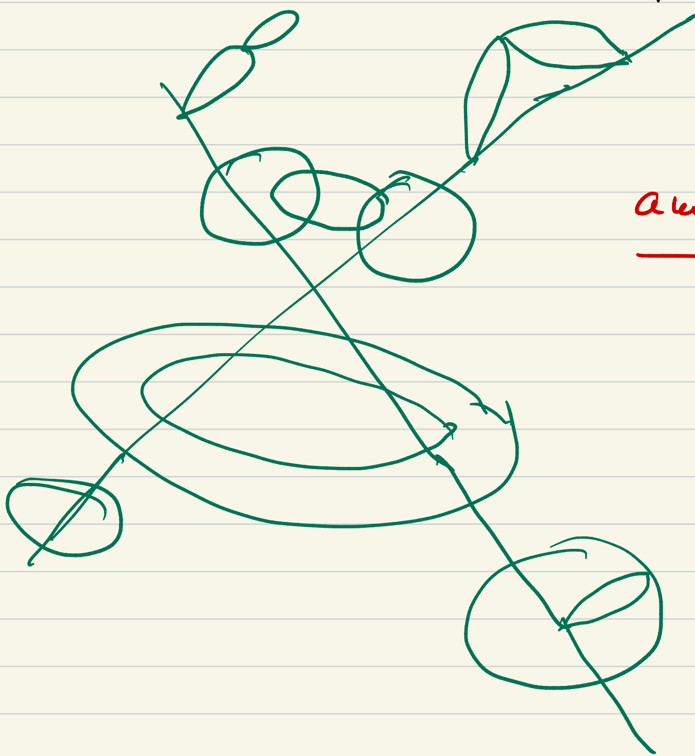


$$a_B^\dagger \, |\Omega\rangle \longrightarrow |p_B\rangle_{1P} + \equiv \left(a_B^{1P\dagger}\right)^\dagger |\Omega\rangle$$

$$a_B^\dagger \, |0\rangle$$

Amputation

$$i\mathcal{M} = \sum \text{all fully connected, amputated diagrams}$$



amputate