Homework 3: Peskin & Schroeder 4.1

Due: Thursday, December 10

Diagramatic technique for ϕ^4

Before: 2-pt Green's fn:

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T \phi_I(x) \phi_I(y) \exp\left[-i \int_{-T}^{T} dt \, H_I(t)\right] | 0 \rangle}{\langle 0 | T \exp\left[-i \int_{-T}^{T} dt \, H_I(t)\right] | 0 \rangle}$$

$$H_I(t) = \frac{\lambda}{4!} \int d^3x \, \phi_I^4(x)$$

Reduced the problem to evaluating expressions of the form:

$$\langle 0 | T \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) | 0 \rangle$$

For $n = 2$ $\qquad \langle 0 | T \phi_I(x) \phi_I(y) | 0 \rangle = D_F(x-y)$

In nonint. theory $\phi_I(x) = \phi(x)$ $\qquad$ Feynmann propagator

$$\langle 0 | T\, \phi_I(x)\, \phi_I(y) | 0 \rangle = D_F(x-y)$$

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$$

$$\phi_I^+(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}}\, a_{\vec{p}}\, e^{-ip\cdot x} \qquad \phi_I^-(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}}\, a^+_{\vec{p}}\, e^{+ip\cdot x}$$

$$\phi_I^+(x)\, |0\rangle \qquad \langle 0 |\, \phi_I^-(x) = 0$$

$x^0 > y^0$ $\qquad T\, \phi_I(x)\, \phi_I(y) =$ normal ordered $+ \left[\phi_I^+(x), \phi_I^-(y)\right]$

$x^0 < y^0$ $\qquad T\, \phi_I(x)\, \phi_I(y) =$ normal ordered $+ \left[\phi_I^+(y), \phi_I^-(x)\right]$

Normal ordered $= N(O) = :O:$ – all $a^+_{\vec{p}}$ (ϕ_I^-) to the left

$\langle 0 | N(O) | 0 \rangle = 0$ $\Longleftarrow$ $\qquad$ of all $a_{\vec{p}}$ (ϕ_I^+)

$x^0 > y^0 \qquad T\,\phi_I(x)\,\phi_I(y) = \text{normal ordered} + [\phi_I^+(x), \phi_I^-(y)]$

$x^0 < y^0 \qquad T\,\phi_I(x)\,\phi_I(y) = \text{normal ordered} + [\phi_I^+(y), \phi_I^-(x)]$

Contraction:

$$\overbracket{\phi_I(x)\,\phi_I(y)} \overset{\text{def}}{=} \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 > y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } y^0 > x^0 \end{cases}$$

From now on drop the I subscript

$$T\,\phi(x)\,\phi(y) = N\left(\phi(x)\,\phi(y) + \overbracket{\phi(x)\,\phi(y)}\right)$$

$$D_F(x-y) = \langle 0|\,T\,\phi(x)\,\phi(y)\,|0\rangle = \overbracket{\phi(x)\,\phi(y)}$$

Generalize to arbitrary # of fields: Wick's theorem

$$T\,\phi(x_1)\cdots\phi(x_n) = N\big(\phi(x_1)\cdots\phi(x_n) + \text{all contractions}\big)$$

Example: $n=4$ $\qquad \phi(x_\alpha) \longrightarrow \phi_\alpha$

$$T\,\phi_1\phi_2\phi_3\phi_4 = N\Big(\phi_1\phi_2\phi_3\phi_4 + \wick{\c{\phi}_1 \c{\phi}_2}\phi_3\phi_4 + \wick{\c{\phi}_1 \phi_2 \c{\phi}_3}\phi_4 +$$

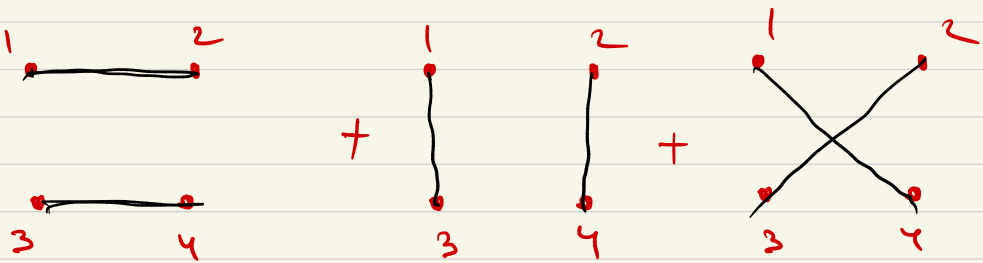
$$+ \wick{\c{\phi}_1 \phi_2 \phi_3 \c{\phi}_4} + \phi_1\wick{\c{\phi}_2 \c{\phi}_3}\phi_4 + \phi_1\wick{\c{\phi}_2 \phi_3 \c{\phi}_4} + \phi_1\phi_2\wick{\c{\phi}_3 \c{\phi}_4} +$$

$$+ \wick{\c1{\phi}_1 \c1{\phi}_2 \c2{\phi}_3 \c2{\phi}_4} + \wick{\c1{\phi}_1 \c2{\phi}_2 \c1{\phi}_3 \c2{\phi}_4} + \wick{\c1{\phi}_1 \c2{\phi}_2 \c2{\phi}_3 \c1{\phi}_4}\Big)$$

$$N\big(\wick{\c{\phi}_1 \phi_2 \c{\phi}_3}\phi_4\big) = D_F(x_1 - x_3)\,N(\phi_2\phi_4)$$

$$\langle 0|T\,\phi_1\phi_2\phi_3\phi_4|0\rangle = D_F(x_1-x_2)\,D_F(x_3-x_4) + $$

$$+\ D_F(x_1-x_3)\,D_F(x_2-x_4) + D_F(x_1-x_4)\,D_F(x_2-x_3)$$



Proof. By induction, $n=2$ ✓

Given $n-1 \longrightarrow n$

Let $x_1^0 \gtrsim x_2^0 \gtrsim \cdots \gtrsim x_n^0$

$$T\phi_1 \cdots \phi_n = \phi_1 \underbrace{\cdots \phi_n} = \underset{\substack{\parallel \\ \phi_1^+ + \phi_1^-}}{\phi_1} N\Big(\phi_2 \cdots \phi_n + \text{all contractions not involving } \phi_1\Big)$$

$$\phi_1^+ N(\phi_2 \cdots \phi_n) = N(\phi_2 \cdots \phi_n)\phi_1^+ + \left[\phi_1^+, N(\phi_2 \cdots \phi_n)\right]$$
$$= N(\phi_1^+ \phi_2 \cdots \phi_n) + \left[\phi_1^+, N(\phi_2 \cdots \phi_n)\right]$$

$$N(c\,\phi_1^+ \phi_2^-) = c\,\phi_2^- \phi_1^+ = c\,N(\phi_2^- \phi_1^+)$$

$$[\phi_1^+, N(\phi_2 \cdots \phi_m)] = N\Big([\phi_1^+, \phi_2^-]\phi_3 \cdots \phi_m +$$

$$+ \phi_2 [\phi_1^+, \phi_3^-]\phi_4 \cdots \phi_m + \cdots \Big)$$

$$T\phi_1 \cdots \phi_m = N(\phi_1 \cdots \phi_m + \text{all contractions})$$

Diagramatics:

points $x_1, \ldots, x_m$ – dots $\bullet$

$D_F(x_i - x_j)$ – lines ———

$$\langle 0 | T \phi_I(x)\, \phi_I(y) \exp\left[-i \int_{-T}^{T} dt\, H_I(t)\right] | 0 \rangle =$$

$$= \langle 0 | T \{ \underbrace{\phi(x)\, \phi(y)}_{\text{free field result}} + \phi(x)\, \phi(y) \underbrace{\left[-i \int dt\, H_I(t) \right]}_{\propto \lambda} + \ldots \} | 0 \rangle$$

$$\text{2nd term} = \langle 0 | T \{ \phi(x)\, \phi(y) \frac{-i\lambda}{4!} \int \underbrace{dt\, d^3z}_{d^4 z} \phi^4 \} | 0 \rangle =$$

$$= \langle 0 | T \{ \phi(x)\, \phi(y) \left(\frac{-i\lambda}{4!} \right) \int d^4 z\, \phi(z)\, \phi(z)\, \phi(z)\, \phi(z) \} | 0 \rangle$$

$\frac{n(n-1)}{2}$? 15 ways to contract

$$\underbrace{\phi \cdots \phi}_{n}$$

$$(n-1)(n-3)\cdots = (n-1)!!$$

$$5 \times 3 \times 1 = 15$$

1st way $\quad \phi(x)\,\phi(y) \quad$ 3

2nd way $\quad \phi(x) \quad \phi(z) \quad$ 4

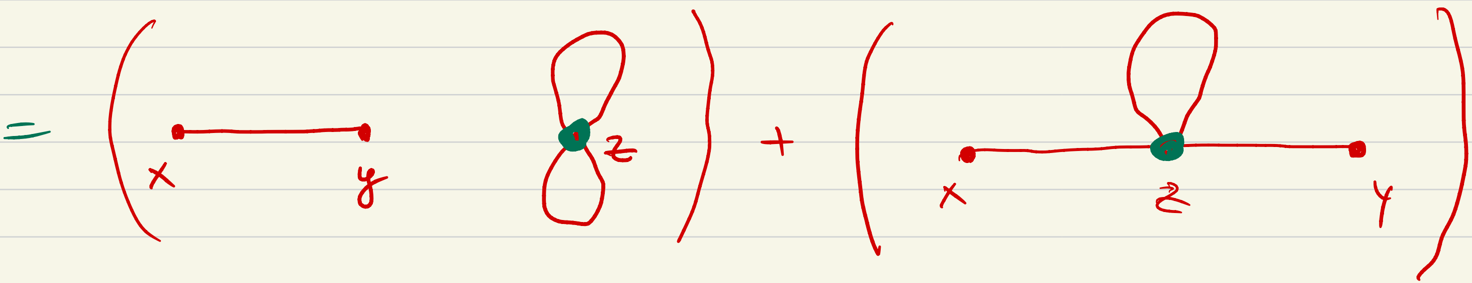
$\phi(y) \quad \phi(z) \quad$ 3

12

$12 + 3 = 15$

2nd term $= 3\left(\frac{-i\lambda}{4!}\right) D_F(x-y)\int d^4z\, D_F(z-z)\, D_F(z-z) +$

$$+\,12\left(-\frac{i\lambda}{4!}\right)\int d^4z\; D_F(x-z)\, D_F(y-z)\, D_F(z-z) =$$



• – vertices

$$\langle 0|\phi(x)\,\phi(y)\,\frac{1}{3!}\left(-\frac{i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi\int d^4w\,\phi\phi\phi\phi\int d^4u\,\phi\phi\phi\phi|0\rangle =$$

$$= \frac{1}{3!}\left(-\frac{i\lambda}{4!}\right)^3\int d^4z\,d^4w\,d^4u\;D_F(x-z)\,D_F(z-z)\,D_F(z-w)\times$$

$$\times\, D_F^2(w-u)\,D_F(w-y)\,D_F(u-u)$$

\# of identical contractions =

$$\left.\begin{array}{ll} 3! & \text{interchange of } z, w, u \text{ vertices} \\ 4\cdot 3 & \text{placements of contract. into } z \\ 4\cdot 3 & \text{placements of contract. into } u \\ 4\cdot 3\cdot 2 & \text{placements of contract. into } w \\ \frac{1}{2} & \text{Interchange of } w-u \text{ contractions} \end{array}\right\} N$$

$$\int d^4 w \, \underset{1}{\phi} \, \underset{2}{\phi} \, \underset{3}{\phi} \, \underset{4}{\phi} \int d^4 z \, \underset{1}{\phi} \, \underset{2}{\phi} \, \underset{3}{\phi} \, \underset{4}{\phi}$$

23 with 12

12 23

$N = 10{,}368$

total # of full contractions of n opts $=$

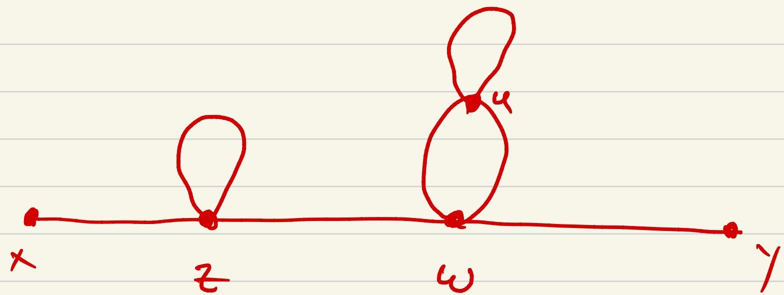
$$= (n-1)(n-3)\cdots 1 = (n-1)!!$$

for $n = 14 \longrightarrow 13!! = 135{,}135$

$$10{,}368 \approx \frac{135{,}135}{13} = 10{,}395$$

$$\frac{1}{3!}\left(-\frac{i\lambda}{4!}\right)^3 \int d^4z\, d^4w\, d^4u \; D_F(x-z)\, D_F(z-z)\, D_F(z-w)\, \times$$

$$\times\, D_F^2(w-u)\, D_F(w-y)\, D_F(u-u)$$



cactus diagram

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