Recall: 2-point correlation fn $\equiv$ 2-pt Green's fn in ϕ^4 theory

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$$

$$H = \underbrace{H_{KG}}_{H_0} + \underbrace{\int d^3x \, \frac{\lambda}{4!}\phi^4(\vec{x})}_{H_{int}}$$

H_{int} enters in two places

1. In $\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}$ – Task #1

2. In $|\Omega\rangle$ – ground state of $H_0 + H_{int}$ – Task #2

$$(H_0 + H_{int})|\Omega\rangle = E_0|\Omega\rangle$$

$H_0|0\rangle = 0$ – defines zero of energy

Interaction picture field:

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)} = \phi(t,\vec{x})\Big|_{\lambda=0}$$

Any difference between $\phi_I(x)$ and $\phi(x)$ is due to H_{int}

$\phi_I(t,\vec{x})$ is the field in the Heisenberg picture in the absence of interactions

$$\phi_I(t,\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)\Big|_{x^0=t-t_0}$$

In terms of $\phi_I(x)$ the full field $\phi(x) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt}$

is:

$$\phi(x) = U^{+}(t, t_0)\,\phi_I(x)\,U(t, t_0)$$

$$U(t, t_0) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$ – interaction picture time-evolution opt

Task #1 is to find $U(t, t_0)$

Last time: $$U(t, t_0) = T\exp\left[-i\int_{t_0}^{t} dt'\, H_I(t')\right]$$

$$H_I(t) = \int d^3x\, \frac{\lambda}{4}\,\phi_I^4(t, \vec{x})$$

Define a more general evolution-type opt:

$$U(t,t') = T \exp\left[-i \int_{t'}^{t} dt'' \, H_I(t'')\right]$$

Last time: $U(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$

Properties of $U(t,t')$:

1. unitary 2. $U^\dagger(t_1,t_2) = U(t_2,t_1)$

3. $U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$

Task #2: Deal with $|\Omega\rangle$ - ground state of $H_0 + H_{int}$

$$|0\rangle = \sum_n \langle n|0\rangle \, |n\rangle$$

eigenstates of H

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$$H|n\rangle = E_n|n\rangle$$

$$Z = \sum_n e^{-E_n/\tau} = e^{-E_0/\tau}\sum_n e^{-\frac{(E_n-E_0)}{\tau}} \xrightarrow[\tau\to 0]{} e^{-E_0/\tau}$$

$$E_0 = -\lim_{\tau\to 0}\left(\tau \ln Z\right)$$

Must <u>assume</u> $\langle \Omega | 0 \rangle \neq 0$

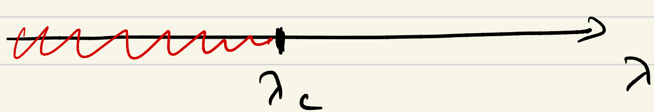
<u>Example:</u> Bardeen – Cooper – Schrieffer (BCS) theory of superconductivity

$$H = \int d^3x \, \psi_\sigma^\dagger \frac{\vec{p}^{\,2}}{2m} \psi_\sigma - \lambda \underbrace{\int d^3x \, \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow}_{\sim \phi^4}$$

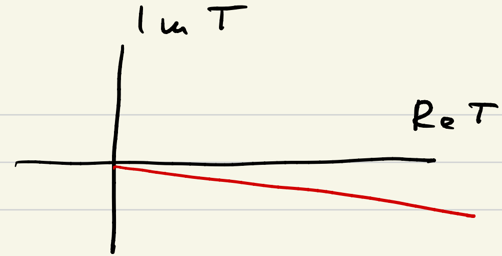
Here $\langle \Omega | 0 \rangle = 0$ in thermodynamic limit

$$E(\lambda) - E(\lambda = 0) = \nu_F \, V \, \Delta^2$$

$$\Delta = \Lambda \, e^{-1/\lambda}$$ – nonanalytic in λ @ $\lambda = 0$ – no expansion in small λ



$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$



$$T \to \infty(1-i\varepsilon)$$

$$e^{iE_0T}e^{-iHT}|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_n e^{-i(E_n-E_0)T}|n\rangle\langle n|0\rangle$$

$$e^{-i(E_n-E_0)T} = e^{-i(\text{real})}\, e^{-\varepsilon(E_n-E_0)\infty} \to 0$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0 T} \langle \Omega | 0\rangle \right)^{-1} e^{-iHT} |0\rangle$$

Shift $T \to T + t_0$

$$e^{-iHT}|0\rangle \longrightarrow e^{-iH(T+t_0)}|0\rangle = \underbrace{e^{-iH(t_0-(-T))}\, e^{-iH_0(-T-t_0)}}_{U^{+}(-T,\,t_0) = U(t_0,\,-T)} |0\rangle$$

$$U(t, t_0) = e^{-iH_0(-t+t_0)}\, e^{-iH(t-t_0)}$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0+T)} \langle \Omega | 0\rangle \right)^{-1} U(t_0,-T) |0\rangle$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle \right)^{-1} U(t_0,-T)|0\rangle$$

Similarly, $$\langle\Omega| = \lim_{T\to\dots} \langle 0|\, U(T,t_0) \left[e^{-iE(T-t_0)}\langle 0|\Omega\rangle \right]^{-1}$$

$$\langle\Omega|T\,\phi(x)\,\phi(y)|\Omega\rangle = \quad \text{Let } x^0 > y^0 > t_0$$

$$= \langle\Omega|\,\phi(x)\,\phi(y)|\Omega\rangle = \lim_{T\to\dots} \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_0T} \right)^{-1} \times$$

$$\times \langle 0|\, \underbrace{U(T,t_0)\; U^{+}(x^0,t_0)}_{U(T,\,x^0)}\, \phi_I(x)\, \underbrace{U(x^0,t_0)\, U^{+}(y^0,t_0)}_{U(x^0,\,y^0)} \times$$

$$\times\, \phi_I(y)\, \underbrace{U(y^0,t_0)\, U(t_0,-T)}_{U(y^0,\,-T)}\, |0\rangle$$

$$1 = \langle \Omega | \Omega \rangle = \lim_{T \to \ldots} \left(|\langle 0 | \Omega \rangle|^2 e^{-2iE_0 T} \right)^{-1} \langle 0 | U(T,-T) | 0 \rangle$$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle =$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T\, U(T,x^0)\, \phi_I(x)\, U(x^0,y^0)\, \phi_I(y)\, U(y^0,-T) | 0 \rangle}{\langle 0 | U(T,-T) | 0 \rangle}$$

$$T \quad \phi_I(x)\, \phi_I(y)\, \underbrace{U(T,x^0)\, U(x^0,y^0)\, U(y^0,-T)}_{U(T,-T)}$$

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | T \phi_I(x) \phi_I(y) \exp\left[-i \int_{-T}^{T} dt' \, H_I(t')\right] | 0 \rangle}{\langle 0 | \exp\left[-i \int_{-T}^{T} dt' \, H_I(t')\right] | 0 \rangle}$$

$$H_I = \int d^3x \, \frac{\lambda}{4!} \phi_I^4(x)$$

Need $\langle 0 | T \, \phi_I(x_1) \, \phi_I(x_2) \cdots \phi_I(x_n) | 0 \rangle$

For $n=2$ – Feynman propagator

$n=2$ $\langle 0|T\,\phi_I(x)\,\phi_I(y)|0\rangle = D_F(x-y)$

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$$

$$\phi_I(t,\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}}\left(\underbrace{a_{\vec{p}}\,e^{-ip\cdot x}}_{\phi_I^+} + \underbrace{a_{\vec{p}}^{+}\,e^{ip\cdot x}}_{\phi_I^-}\right)$$

$$\phi_I^+(x)|0\rangle = 0$$

$$\langle 0|\,\phi_I^-(x) = 0$$

$$\phi_I^-\,\phi_J^-\;\cdots\;\phi_I^+\,\phi_J^+$$

Let $x^0 > y^0$

$$T\phi_I(x)\,\phi_I(y) = \phi_I(x)\,\phi_I(y) = \left(\phi_I^+(x) + \phi_I^-(x)\right)\left(\phi_I^+(y) + \phi_I^-(y)\right) =$$

$$= \phi_I^+(x)\,\phi_I^+(y) + \underline{\phi_I^+(x)\,\phi_I^-(y)} + \phi_I^-(x)\,\phi_I^+(y) + \phi_I^-(x)\,\phi_I^-(x) =$$

$$= \phi_I^+(x)\,\phi_I^+(y) + \underline{\phi_I^-(y)\,\phi_I^+(x)} + \phi_I^-(x)\,\phi_I^+(y) + \phi_I^-(x)\,\phi_I^-(x) +$$

$$+ \underline{\left[\phi_I^+(x), \phi_I^-(y)\right]}$$

Normal order: $a^+_{\vec{p}}\, a^+_{\vec{q}}\, a_{\vec{k}}$ – normal ordered

$a^+\, a\, a^+$ – not

Normal ordering opt

$$N\left(a_{\vec{p}}\, a^\dagger_{\vec{k}}\, a_{\vec{q}}\right) = a^\dagger_{\vec{k}}\, a_{\vec{p}}\, a_{\vec{q}}$$

$$N\left(\underline{\quad}\right) = :\underline{\quad}:$$

For $y^0 > x^0$ $\quad [\phi_I^+(y), \phi_I^-(x)]$

$$\overline{\phi(x)\,\phi(y)} = \begin{cases} \underline{[\phi_I^+(x), \phi_I^-(y)]}, & x^0 > y^0 \\ [\phi_I^+(y), \phi_I^-(x)], & y^0 > x^0 \end{cases}$$

$$\parallel$$

$$D_F(x-y)$$

$$T\,\phi(x)\,\phi(y) = N\left(\phi(x)\,\phi(y) + \overbrace{\phi(x)\,\phi(y)}\right)$$