CM $\longrightarrow$ QM $\longrightarrow$ relativistic

$$S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t) \longrightarrow \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$x^\mu = (t, \vec{x})$$

$$\phi(x^\mu)$$

Least action principle: S = extremum for fixed t_1 & t_2

$$\phi \to \phi + \delta\phi \qquad \delta S = 0$$

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

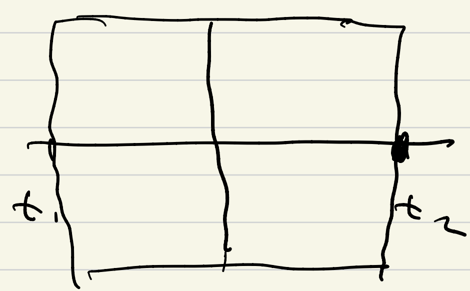
$$\phi \to \phi + \delta\phi$$

$$\partial_\mu \phi \to \partial_\mu \phi + \partial_\mu (\delta\phi)$$

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi) \right\}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi$$

$$\delta\phi \Big|_{\text{boundary}} = 0$$



$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

Euler-Lagrange eqs. for a field

Hamiltonian: conjugate momentum $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

$$H = \sum_i p_i \dot{q}_i - L$$

$$p(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3y \, \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y})) =$$

$$\sum_{\vec{y}} \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y})) d^3y$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} d^3x$$

Momentum density $\quad \pi(\vec{x}) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})}$

$$H = \sum_{\vec{x}} p(\vec{x})\, \dot{\phi}(\vec{x}) - L =$$

$$= \int d^3x \left[\pi \dot{\phi} - \mathcal{L} \right]$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

Hamiltonian density

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x-y)$$

Ex.

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)$$

ϕ – real

$$x^\mu = (t, \vec{x}) \qquad x_\mu = (t, -\vec{x})$$

$$x_\mu = g_{\mu\nu}\, x^\nu \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\, \partial^\mu\phi - m^2\phi^2\right)$$

$$(\partial_\mu\phi)^2$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right) \qquad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$\left(\partial_t^2 - \nabla^2 + m^2 \right) \phi = 0$$

Klein – Gordon eq.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial^t \phi = \dot{\phi}$$

$$\mathcal{H} = \dot{\phi}^2 - \mathcal{L} = \frac{\pi^2}{2} + \frac{(\nabla \phi)^2}{2} + \frac{m^2 \phi^2}{2}$$

shear

Noether thm

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

Symmetry if S = invariant

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\, \partial_\mu J^\mu(x)$$

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \Delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\alpha \Delta \phi) =$$

$$= \alpha\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \Delta\phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta\phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) = \partial_\mu J^\mu$$

$$\partial_\mu j^\mu = 0 \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$$

j^μ – conserved current

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

$$Q = \int_{\text{all space}} j^0 \, d^3x = \text{const in time}$$

Ex. 1

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$$

$$\phi \longrightarrow \phi + \alpha \qquad \mathcal{L} = \text{const}$$

$$J^\mu = 0$$

$$j^\mu = \partial^\mu \phi \qquad \int \dot{\phi}\, d^3x = \left(\int \phi\, d^3x\right)^{\cdot} \qquad \Delta\phi = 1$$

Ex. 2

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

$$\phi \in \mathbb{C}$$

$$\mathcal{L} = \text{inv} \quad \text{for} \quad \phi \to e^{i\alpha}\phi \qquad e^{i\alpha} \approx 1 + i\alpha$$

$$\alpha\Delta\phi = i\alpha\phi \qquad \alpha\Delta\phi^* = -i\alpha\phi^*$$

Noether current

$$J^\mu = i\left[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)\right]$$

$$\frac{\partial\, J^{t*}}{\partial t} + \nabla \cdot \vec{J} = 0$$

spacetime transform. — translation & rotations

$$x^\nu \to x^\nu - a^\nu \qquad 4 \text{ symm.}$$

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\nu \partial_\nu \phi(x)$$

$$\mathcal{L} \to \mathcal{L} + a^\nu \partial_\nu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$$

$$j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\,\delta^\mu_\nu$$

$$\equiv$$

$$T^\mu{}_\nu$$

stress-energy tensor

$$T^0{}_i = \frac{\partial \mathcal{L}}{\partial \dot\phi} \partial_i \phi = \pi\, \partial_i \phi$$

$$T^{0i} = -\pi\, \partial_i \phi$$

$$H = \int T^{00}\, d^3x = \int \mathcal{H}\, d^3x$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = \mathcal{H}$$

$$P^i \equiv \int T^{0i}\, d^3x = -\int \pi\, \partial_i \phi\, d^3x$$