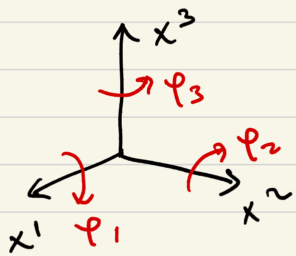
Spin of Dirac particle

Angular momentum of Dirac particle = ?

Recall: CM: isotropic $\Rightarrow$ $\vec{\ell}$ = const



$$\frac{\partial L}{\partial \varphi_i} = \frac{\partial H}{\partial \varphi_i} = 0 \qquad \ell_i = \frac{\partial L}{\partial \dot{\varphi}_i} = \text{const}$$

$$\{\vec{\ell}, H\}_P = 0$$

QM: $\vec{\ell} \to \hat{\vec{\ell}}$ – generator of rotations $\quad R = e^{-i\vec{\alpha}\cdot\vec{\ell}}$

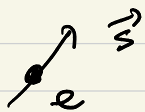
$$[\hat{\vec{\ell}}, H] = 0 \Rightarrow \hat{\ell}_i \text{ – conserved}$$

$\uparrow$ rotational inv $\qquad$ good quantum #

$H \to R^{-1} H R$

Magnetic moment $\quad \hat{\vec{\mu}}_e = \frac{q}{2m} \hat{\vec{\ell}}$ (can get this from CM)

Spin enters

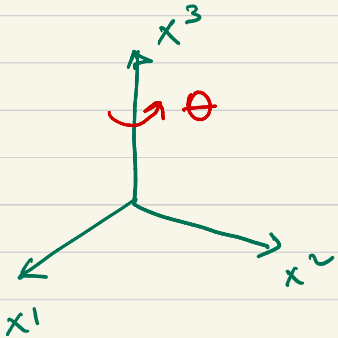


$$\vec{\mu}_s = g \frac{q}{2m} \vec{S}$$

$$g_e \approx 2$$

QFT resolves this mystery.

Noether thm



$$\begin{pmatrix}\tilde{x}^1\\ \tilde{x}^2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x^1\\ x^2\end{pmatrix}$$

$$\simeq$$

$$\begin{pmatrix}1 & \theta\\ -\theta & 1\end{pmatrix} = \Lambda^{-1}$$

$$\tilde{x}^1 = x^1 + \theta x^2 \qquad \tilde{x}^2 = x^2 - \theta x^1$$

$$\psi(x) \longrightarrow \tilde{\psi}(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x) \qquad \mathcal{L} \to \tilde{\mathcal{L}}$$

$$\Lambda_{1/2} = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}$$ In our case $\omega_{12} = -\omega_{21} = \theta$

$$S^{12} = \frac{1}{2}\Sigma^3 \qquad \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\Lambda_{1/2} = 1 - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu} = 1 - \frac{i}{2}\theta\,\Sigma^3$$

$$\tilde{\psi}(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x)$$

$$\psi(\Lambda^{-1}x) = \psi(x^0, x^1 + \theta x^2, x^2 - \theta x^1, x^3) \simeq \psi(x) + \theta x^2 \partial_1 \psi -$$
$$- \theta x^1 \partial_2 \psi = (1 + \theta x^2 \partial_1 - \theta x^1 \partial_2)\psi$$

$$\delta\psi = \tilde{\psi} - \psi = \left(1 - \frac{i}{2}\theta\,\Sigma^3\right)(1 + \theta x^2\partial_1 - \theta x^1 \partial_2)\psi - \psi$$

$$\delta \psi = \tilde{\psi} - \psi = \underbrace{\left(1 - \frac{i}{2}\theta \Sigma^3\right)\left(1\right.}\left. + \theta x^2 \partial_1 - \theta x^1 \partial_2\right)\psi - \psi =$$

$$= \theta \left[x^2 \partial_1 - x^1 \partial_2 - \frac{i}{2} \Sigma^3 \right] \psi$$

Noether thm: $\psi \to \psi + \underbrace{\alpha \Delta \psi}_{\delta \psi}$ $\quad \mathcal{L} \to \mathcal{L}$

$$\Rightarrow \quad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi \quad : \quad \partial_\mu J^\mu = 0$$

$$\Rightarrow \quad Q = \int_{\text{all space}} J^0 \, d^2x \; - \text{conserved}$$

$$\alpha \to \theta$$

$$\Delta\psi = \left[x^2\partial_1 - x^1\partial_2 - \frac{i}{2}\Sigma^3 \right]\psi \qquad Q = \int_{\text{all space}} J^0\, d^3x$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi \qquad \mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$$

$$\psi^\dagger \underbrace{\gamma^0 \quad \gamma^0}_{=1} \partial_0\psi$$

$$J^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = i\,\psi^\dagger\left[x^2\partial_1 - x^1\partial_2 - \frac{i}{2}\Sigma^3\right]\psi$$

$$\left[\vec{x}\times(-i\nabla)\right]_3$$

$$\Rightarrow \vec{J} = \int d^3x\ \psi^\dagger\left[\vec{x}\times(-i\nabla) + \frac{1}{2}\vec{\Sigma}\right]\psi$$

$$\vec{J} = \int d^3x \; \psi^+ \Big[\underbrace{\vec{x} \times (-i \nabla)} + \underbrace{\tfrac{1}{2} \vec{\Sigma}} \Big] \psi = \text{const}$$

nonrelativistic QM: $\hat{\vec{\ell}} + \hat{\vec{s}} = \hat{\vec{J}}$

Note: $\vec{J} = \int d^3x \; \psi^+ \vec{J} \, \psi$

zero momentum fermion / antifermion

$a_0^{s+} |0\rangle \qquad b_0^{s+} |0\rangle$

Need: $J_z \, a_0^{s+} |0\rangle \qquad J_z \, b_0^{s+} |0\rangle$

Need: $J_z\, a_0^{s\dagger} |0\rangle \qquad J_z\, b_0^{s\dagger} |0\rangle$

1. Heisenberg $\to$ Schrödinger

$$\psi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}}\, v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right]$$

2. Choose

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}$$

Expect

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$J_z \, a_0^{s+} |0\rangle = \pm \frac{1}{2} a_0^{s+} |0\rangle$ $\qquad$ $J_z \, b_0^{s+} |0\rangle = \mp \frac{1}{2} b_0^{s+} |0\rangle$

upper sign $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ lower sign $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$j^\mu = \bar{\psi} \gamma^\mu \psi$ — conserved $\qquad j^0 = \psi^\dagger \gamma^0 \gamma^0 \psi = \psi^\dagger \psi$

$$Q = \int d^3x \; \psi^\dagger(x) \psi(x) = \int_{\vec{p}} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right)$$

$$b b^\dagger = 1 - b^\dagger b$$

$$Q = \int_{\vec{p}} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

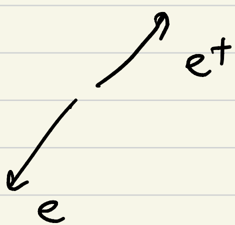
$a^{s\dagger}_{\vec{p}}$ — fermion with charge $+1$

$b^{s\dagger}_{\vec{p}}$ — antifermion with charge -1

In QED

$a_p^{s\dagger}$ – electron with energy $E_{\vec{p}}$, momentum $\vec{p}$, spin-1/2 polarization ξ^s, charge $-e$

$b_p^{s\dagger}$ – positron, $E_{\vec{p}}$, $\vec{p}$, spin-1/2 polarization opposite to ξ^s, charge $+e$



$\psi(x)|0\rangle$ positron @ x with ξ^s

$\bar{\psi}(x)|0\rangle$ electron @ x ....