Quantization of the Dirac field $\psi(x)$

1. Expand in single-particle eigenstates

$$\psi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}}\, v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\bar{\psi}(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}}\, \bar{v}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right]$$

$$H = \int_{\vec{p}} \sum_s \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}}\, b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

2. Promote a^s_b and b^s_p to operators & postulate comm. relations

Trouble: with canonical comm. relations can lower the energy indefinitely by creating more & more b-particles

Causality:

$$[\psi(x), \bar{\psi}(y)] = \langle \psi(x)\bar{\psi}(y)\rangle - \langle \bar{\psi}(y)\psi(x)\rangle$$

vanishes for spacelike $(x-y)$ — vanishes identically

In KG $$[\varphi(x), \varphi(y)] = \langle \varphi(x)\varphi(y)\rangle - \langle \varphi(y)\varphi(x)\rangle$$

$$\langle \varphi(x)\varphi(y)\rangle = \langle \varphi(y)\varphi(x)\rangle$$

particle $y \to x$ — antiparticle $x \to y$

Can we have the same cancelation for Dirac?

i.e. $\langle \psi(x)\,\bar{\psi}(y)\rangle = \langle \bar{\psi}(y)\,\psi(x)\rangle$

a-particle $\quad$ b-particle

$y \to x$ $\quad$ $x \to y$

Need: $b^{s\dagger}_{\vec{p}}\,|0\rangle = 0, \quad b^{s}_{\vec{p}}\,|0\rangle \neq 0$

for a-particle as usual $a^{s\dagger}_{\vec{p}}\,|0\rangle \neq 0, \quad a^{s}_{\vec{p}}\,|0\rangle = 0$

$$\psi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p})\, e^{i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}}\, v^s(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\bar{\psi}(\vec{y}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p})\, e^{-i\vec{p}\cdot\vec{y}} + b^{s\dagger}_{\vec{p}}\, \bar{v}^s(\vec{p})\, e^{i\vec{p}\cdot\vec{y}} \right]$$

Then, using symmetry arguments, we found

$$\langle \psi(x)\,\bar\psi(y)\rangle = +(i\not\partial_x + m)\int_{\vec p}\frac{e^{+ip\cdot(x-y)}}{2E_{\vec p}} \qquad A$$

$$\langle \bar\psi(y)\,\psi(x)\rangle = -(i\not\partial_x + m)\int_{\vec p}\frac{e^{-ip\cdot(x-y)}}{2E_{\vec p}} \qquad B$$

$e^{iE_p t}\quad e^{-i\vec p\cdot\vec x}$

$\|$

$\cos(E_p t)$

$i\sin(E_p t)$

Aso deska

$A, B > 0$

For spacelike $(x-y)$ go to ref. frame where $x_0 - y_0 = 0$

and $\vec p \to -\vec p$ under B-integral

$\Rightarrow$ need $A = -B$ for $[\psi(x), \bar\psi(y)]$ to vanish

But this is impossible b/c $A, B > 0$

Let $A = B = 1$

$$\Rightarrow \quad \langle \psi(x) \bar{\psi}(y) \rangle = - \langle \bar{\psi}(y) \psi(x) \rangle \quad \text{for spacelike } (x-y)$$

Spinor fields anticommute at spacelike separation.

This is enough to preserve causality b/c observables (energy, charge, particle #) are built out of even # of spinor fields

$$[O_1(x), O_2(y)] = 0$$

$$\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) = \bar{\psi}(y)\psi(y)\bar{\psi}(x)\psi(x)$$

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$\{A, B\} = AB + BA$

$[A, B] = AB - BA$

$$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})\} = 0$$

$\bar{\psi} = \psi^\dagger \gamma^0$

$$\Updownarrow$$

$$\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}$$

$$\{a_{\vec{p}}^r, a_{\vec{q}}^s\} = \{b_{\vec{p}}^r, b_{\vec{q}}^s\} = 0$$

$$H = \int_{\vec{p}} \sum_s \left(E_{\vec{p}}\, a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s - E_{\vec{p}}\, b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s \right)$$

$$-E_{\vec{p}}\, b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s = E_{\vec{p}}\, b_{\vec{p}}^s\, b_p^{s\dagger}$$

$b_p^{s\dagger} = \tilde{b}_p^s$

$b_p^s = \tilde{b}_p^{s\dagger}$

Redef $|0\rangle$: $\quad a^s_{\vec{p}}\,|0\rangle = \tilde{b}^s_{\vec{p}}\,|0\rangle = 0$

Consider b & b^+ such that $\{b, b^+\} = 1$

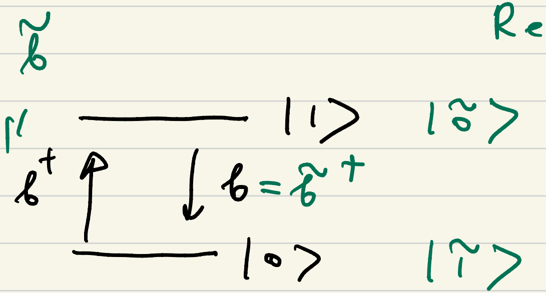
$$\{b, b\} = \{b^+, b^+\} = 0$$

Def $|0\rangle$: $\quad b|0\rangle = 0 \qquad |1\rangle \equiv b^+|0\rangle \qquad b|1\rangle = |0\rangle$?

$$b^+|1\rangle = (b^+)^2|0\rangle = 0$$

$$(b b^+ + \cancel{b^+ b})|0\rangle = |0\rangle$$

$$b|1\rangle = |0\rangle$$



Redefine: $\quad \tilde{b} = b^+$

$$\tilde{b}^+ = b$$

$$\tilde{b}^+|\tilde{0}\rangle = |\tilde{1}\rangle$$

$$\tilde{b}|\tilde{1}\rangle = |\tilde{0}\rangle$$

$$\tilde{b}|\tilde{0}\rangle = 0$$

$$(\tilde{b}^+)^2|\tilde{0}\rangle = 0$$

Note $\quad a^{s\dagger}_{\vec{p}}\, a^{r\dagger}_{\vec{q}}\, |0\rangle = -\, a^{r\dagger}_{\vec{q}}\, a^{s\dagger}_{\vec{p}}\, |0\rangle$

Multiparticle state is antisymm. w.r.t. interchange of 2 particles $\Rightarrow$ Fermi - Dirac statistics

More general result: W. Pauli, Phys. Rev. 58, 718 (1940)

Lorentz inv, positive energies & norms, causality

$\Rightarrow$ Spin = integer Bose - Einstein

Spin = integer + 1/2 Fermi - Dirac

$$\tilde{b} \to b \qquad \psi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}}\, v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\bar{\psi}(\vec{y}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{y}} + b^s_{\vec{p}}\, \bar{v}^s(\vec{p}) e^{i\vec{p}\cdot\vec{y}} \right]$$

$$H = \int_{\vec{p}} \sum_s \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + E_{\vec{p}}\, b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

Vacuum: $\quad a^s_{\vec{p}} |0\rangle = b^s_{\vec{p}} |0\rangle = 0$

$$\vec{P} = \int d^3x\, \psi^\dagger (-i\nabla) \psi = \sum_s \int_{\vec{p}} \vec{p} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$$

$a_{\vec{p}}^{s\dagger}$ & $b_{\vec{p}}^{s\dagger}$ create particles with energy $E_{\vec{p}}$

$\hookrightarrow$ fermions

$\hookrightarrow$ antifermions

1-particle states $\quad |\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, a_p^{s\dagger} |0\rangle$

$$\langle \vec{p}, r | \vec{q}, s\rangle = 2E_p \underset{\uparrow}{(2\pi)^3}\, \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}$$

Lorentz inv

Spin of Dirac particle

$$\psi(x) \to \psi'(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x) \qquad \mathcal{L} \to \mathcal{L}$$

Λ – rotation by θ (= small) around z-axis

Noether thm: $$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\Delta\psi \qquad \Delta\psi = \psi' - \psi$$

$$\{\psi, \psi^{+}\} = 0 \Rightarrow \{\psi_a, \psi_b^{*}\} = 0$$

$$\bar{\psi} = \psi^{+} \gamma^{0}$$

$$\bar{\psi}_c = \psi_b^{*} (\gamma^{0})_{bc}$$

$$\{\psi_a, \bar{\psi}_c\} = \{\psi_a, \psi_b^{*}\} \gamma^{0}_{bc} = 0$$

$$\Downarrow$$

$$\{\psi, \bar{\psi}\} = 0$$