Quantization of the Dirac field $\psi(x)$

1. Expand in single-particle eigenstates

$$\psi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}}\, v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$\bar{\psi}(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b^{s\dagger}_{\vec{p}}\, \bar{v}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right]$$

$$H = \int_{\vec{p}} \sum_s \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}}\, b^{s\dagger}_{\vec{p}} b_{\vec{p}} \right)$$

2. Promote a^s_b and b^s_p to operators & postulate comm. relations

Trouble: with canonical comm. relations can lower the energy indefinitely by creating more & more b-particles

Causality - switch to Heisenberg

$$e^{iHt} a^s_{\vec{p}} e^{-iHt} = a^s_{\vec{p}} e^{-iE_{\vec{p}} t}$$

$$e^{iHt} b^s_{\vec{p}} e^{-iHt} = b^s_{\vec{p}} e^{iE_{\vec{p}} t}$$

$$\psi(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + b^s_{\vec{p}} v^s(\vec{p}) e^{ip\cdot x} \right]$$

$$\bar{\psi}(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}} \bar{u}^s(\vec{p}) e^{ip\cdot x} + b^{s\dagger}_{\vec{p}} \bar{v}^s(\vec{p}) e^{-ip\cdot x} \right]$$

$$[\psi(x), \bar{\psi}(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0$$

$$\not{\partial} = \partial_\mu \partial^\mu$$

$$[\psi(x), \bar{\psi}(y)] = (i\not{\partial}_x + m)\boxed{\int_{\vec{p}} \frac{e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}}{2E_{\vec{p}}}}$$

$$= [\phi(x), \phi(y)]$$

$$[\psi(x), \bar{\psi}(y)] = \langle 0|[\psi(x), \bar{\psi}(y)]|0\rangle = \qquad \cancel{\langle y|x\rangle}$$

$$= \langle 0|\psi(x)\bar{\psi}(y)|0\rangle - \langle 0|\bar{\psi}(y)\psi(x)|0\rangle$$

$$\psi(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p})\, e^{-ip\cdot x} + b^s_{\vec{p}}\, v^s(\vec{p})\, e^{ip\cdot x}\right]$$

$$\bar{\psi}(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^{s\dagger}_{\vec{p}}\, \bar{u}^s(\vec{p})\, e^{ip\cdot x} + b^{s\dagger}_{\vec{p}}\, \bar{v}^s(\vec{p})\, e^{-ip\cdot x}\right]$$

Recall Klein – Gordon

$$\varphi^{+} = \varphi$$

$$[\varphi(x), \varphi(y)] = 0 \quad b/c$$

$$\langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \langle 0 | \varphi(y) \varphi(x) | 0 \rangle$$

amp. for part $y \to x$ antipart $x \to y$

$$\psi(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_s \left[a^s_{\vec{p}}\, u^s(\vec{p})\, e^{-ip\cdot x} + b^s_{\vec{p}}\, v^s(\vec{p})\, e^{ip\cdot x} \right]$$

$$\langle 0 | \psi(x)\, \bar{\psi}(y) | 0 \rangle = \langle 0 | \int_{\vec{p}} \frac{1}{\sqrt{2E_p}} \sum_r a^r_p\, u^r(p)\, e^{-ip\cdot x} \times$$

$$\int_{\vec{q}} \frac{1}{\sqrt{2E_q}} \sum_s a^{s\dagger}_q\, \bar{u}^s(q)\, e^{+iq\cdot y} | 0 \rangle$$

$$\langle 0 |\, a^r_p\, a^{s\dagger}_q \,| 0 \rangle$$

$$\langle 0 | a_p^r \, a_q^{s\dagger} | 0 \rangle$$

translational & rotational symm on $|0\rangle$

$$e^{i\hat{\vec{P}}\cdot\vec{x}} |0\rangle = |0\rangle$$

$$\langle 0 | a_{\vec{p}}^r \, a_{\vec{q}}^{s\dagger} e^{i\hat{\vec{P}}\cdot\vec{x}} | 0 \rangle = e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \langle 0 | e^{i\hat{\vec{P}}\cdot\vec{x}} a_{\vec{p}}^r \, a_q^{s\dagger} | 0 \rangle$$

$$\langle 0 | a_{\vec{p}}^r \, a_{\vec{q}}^{s\dagger} | 0 \rangle = e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \langle 0 | a_{\vec{p}}^r \, a_{\vec{q}}^{s\dagger} | 0 \rangle$$

$$\overset{\parallel}{\langle \cdots \rangle} \qquad \langle \cdots \rangle = 0 \quad \text{for} \quad \vec{q} \neq \vec{p}$$

similarly rot. inv. $\Longrightarrow$ $r = s$

$$\langle 0 | a^{r}_{\vec{p}} \, a^{s\dagger}_{\vec{q}} | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \, \delta_{rs} \, A(\vec{p})$$

need $A(\vec{p}) > 0$ for the norm > 0

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{1}{2E_{\vec{p}}} \underbrace{\sum_s u^s(p) \, \bar{u}^s(p)}_{\gamma \cdot p + m \; = \; \not{p} + m} A(\vec{p}) \, e^{-ip\cdot(x-y)} =$$

$$= \int \frac{1}{2E_{\vec{p}}} (\not{p} + m) \, A(\vec{p}) \, e^{-ip\cdot(x-y)}$$

$p^2 = m^2$

$$A(\vec{p}) = A(p^2) = \text{const}$$

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = (i\not{\partial}_x + m)\int_{\vec{p}} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \quad A \quad (\square)$$

$$\to e^{-iE_p\Delta t} e^{i\vec{p}\cdot\Delta\vec{x}}$$

$\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$ – want only b-particles

$$\Rightarrow \langle 0|\bar{\psi}(y)\psi(x)|0\rangle = -(i\not{\partial}_x + m)\int_{\vec{p}} \frac{1}{2E_{\vec{p}}} e^{+ip\cdot(x-y)} B \quad (\square\square)$$

$B > 0$

$[\psi(x), \bar{\psi}(y)] = 0$ need $\square = \square\square$

$\Rightarrow A = -B$

$$\| \quad \psi(x)\bar{\psi}(y) - \bar{\psi}(y)\psi(x)$$

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle + \langle 0|\bar{\psi}(y)\psi(x)|0\rangle = 0$$

Set $A = B = 1$

$$\{f(x), \bar{F}(y)\} = 0 = f(x)\bar{F}(y) + \bar{F}(y)f(x)$$