Dirac field bilinear $\qquad \bar{\psi} = \psi^\dagger \gamma^0$

$\bar{\psi}\,\Gamma\,\psi \in \mathbb{R}$

$$(\bar{\psi}\,\Gamma\,\psi)^\dagger = \bar{\psi}\,\Gamma\,\psi = \psi^\dagger \gamma^0 \Gamma \psi$$

$$\shortparallel$$

$$(\psi^\dagger \gamma^0 \Gamma \psi)^\dagger = \psi^\dagger (\gamma^0 \Gamma)^\dagger \psi$$

$(\gamma_0 \Gamma)^\dagger = \gamma_0 \Gamma \qquad \gamma_0 \Gamma$ – Hermitian

$\Gamma = \gamma_0 M \qquad \gamma_0 \Gamma = M$

$\bar{\psi}\psi$ - scalar $\qquad \bar{\psi}\gamma^\mu\psi$ - vector

$\gamma^0\gamma^\mu$ - Hermitian

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$(\gamma^0)^+ = \gamma^0 \qquad \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \text{ - Hermitian}$$

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	2nd rank tensor	6
$\gamma^\mu \gamma^5$	(pseudo) vector	4
γ^5	(pseudo) scalar	1
		16

$$\vec{S} = \frac{\vec{\sigma}}{2}$$

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{\mu\nu\rho\sigma} = -i\ \varepsilon^{\mu\nu\rho\sigma}\,\gamma^5$$

4! terms

Ex.

$$\gamma^{1023} = \frac{1}{4!}\left(\gamma^1\gamma^0\gamma^2\gamma^3 - \gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^3\gamma^2 + \dots\right. =$$

$$= \gamma^1\gamma^0\gamma^2\gamma^3 = \underbrace{\varepsilon^{1023}}_{=-1}\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{\mu\nu\rho} = +i\,\varepsilon^{\mu\nu\rho\sigma}\,\gamma_\sigma\,\gamma^5$$

$$\gamma^{123} = +i\,\varepsilon^{1230}\,\underbrace{\gamma_0\,(\,i\,\gamma^0}_{i}\,\gamma^1\gamma^2\gamma^3\,) = +\gamma^1\gamma^2\gamma^3$$

$$\gamma^{123} = \gamma^1\gamma^2\gamma^3$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma^5)^\dagger = \gamma^5 \qquad (\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

$$i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -\gamma^\mu\gamma^5$$

$$\overset{\shortparallel}{\gamma^5\gamma^\mu}$$

$$\gamma^5 = i\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\downarrow$

diagonal

γ^5 – (pseudo)scalar

$$J^\mu = \bar{\psi} \gamma^\mu \psi \qquad \bar{\psi} \gamma^\mu \gamma^5 \psi = J^{\mu 5}$$

$$\partial_\mu J^\mu = (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu (\partial_\mu \psi) =$$

$$i \gamma^\mu \partial_\mu \psi = m \psi$$

$$= i m \bar{\psi} \psi + \bar{\psi} (-i m \psi) = 0$$

$$-i \partial_\mu \bar{\psi} \gamma^\mu = m \bar{\psi}$$

$$\partial_\mu J^{\mu 5} = (i m \bar{\psi}) \gamma^5 \psi - \bar{\psi} \gamma^5 \gamma^\mu \partial_\mu \psi =$$

$$= (i m \bar{\psi}) \gamma^5 \psi + \bar{\psi} \gamma^5 (i m \psi) =$$

$$= 2 i m \bar{\psi} \gamma^5 \psi$$

j^μ – electric charge current density

(1) $\psi(x) \to e^{i\alpha}\psi(x)$ (1)

$(1+i\alpha)$

$\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$ (2)

$(1+i\alpha\gamma^5)$

chiral transformation

$$\mathcal{L}_{Dirac} = \bar{\psi}\, i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

(1) is symm. of $\mathcal{L}_{Dirac}$

(2) only if $m=0$

$$\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 =$$

$$\bar{\psi}\, e^{i\alpha\gamma^5}$$

$$\bar\psi\psi \to \bar\psi e^{i\alpha\gamma^5} e^{i\alpha\gamma^5}\psi = \bar\psi e^{2i\alpha\gamma^5}\psi \neq \bar\psi\psi$$

$$\bar\psi i\gamma^\mu\partial_\mu\psi \longrightarrow \bar\psi \cancel{e^{i\alpha\gamma^5}}\, i\gamma^\mu\, e^{i\alpha\gamma^5}\partial_\mu\psi = \text{inv}$$

$$\cancel{e^{-i\alpha\gamma^5}}$$

$$\mathcal{L}_{Dirac} = \bar\psi i\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$$

$$\mathcal{L}_{Dirac} \to \mathcal{L}_{Dirac} + m\left(\bar\psi\psi - \bar\psi \underbrace{e^{2i\alpha\gamma^5}}_{= 1+2i\alpha\gamma^5}\psi\right)$$

$$-\alpha\left[2i\,\bar\psi\gamma^5\psi\right] \qquad \partial_\mu a^\mu$$

Recall: Noether thm

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$$\psi \to \psi + \alpha \Delta\psi \qquad \mathcal{L} \to \mathcal{L} + \alpha \partial_\mu J^\mu$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi - J^\mu \qquad \text{conserved}$$

(1) $\psi \to (1 + i\alpha)\psi$

$$\Delta\psi = i\psi$$

(2) $\psi \to (1 + i\alpha\gamma^5)\psi$

$$\Delta\psi = i\gamma^5\psi$$

(1) $$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi = \bar{\psi} i\gamma^\mu (i\psi) = -\bar{\psi}\gamma^\mu\psi$$

$$\mathcal{L}_{Dirac} = \bar{\psi} \, i \gamma^\mu \partial_\mu \psi$$

(2)

$$j^{\mu 5} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi = \bar{\psi} i \gamma^\mu \, (i \gamma^5 \psi) = -\bar{\psi}\gamma^\mu\gamma^5\psi$$

$$\int d^4x \, \bar{\psi}\psi = \int d^4x \; \partial_\mu J^\mu = \int ds_\mu \; J^\mu$$

$$\partial_\mu J^\mu = \bar{\psi}\psi$$