Previously: Free particle solutions of the Dirac eq.

$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \psi(x) = u(p)\, e^{-ip\cdot x}$ where $p^2 = m^2$

Took $p = (E_p, \vec{p})$, i.e. $p^0 > 0$

There are also negative freq. solutions $v(p)\, e^{+ip\cdot x}$ $(-p)^2 = m^2$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$u^\dagger(p) = \left(\xi^\dagger \sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)$$

$$\bar u(p) = u^\dagger(p)\gamma^0 = \left(\xi^\dagger\sqrt{p\cdot\bar\sigma},\ \xi^\dagger\sqrt{p\cdot\sigma}\right)$$

$$\sigma = (\mathbb{1}, \vec\sigma),\quad \bar\sigma = (\mathbb{1}, -\vec\sigma)$$

Norm: $u^\dagger u = 2E_p\,\xi^\dagger\xi$ — not Lorentz inv

$$\bar u\, u = 2m\,\xi^\dagger\xi$$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \leftarrow \boxed{\text{2 indep. solutions}}$$

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$$\bar u u = 2m\,\xi^\dagger\xi$$

$$\xi^1 = \begin{pmatrix}1\\0\end{pmatrix},\quad \xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \leftarrow \boxed{\text{2 indep. solutions}}$$

$$u^{s\dagger}(p) = \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\right)$$

$$\bar u^s(p) = u^{s\dagger}(p)\gamma^0 = \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$$\sigma = (\mathbb{1}, \vec\sigma),\quad \bar\sigma = (\mathbb{1}, -\vec\sigma)$$

Normalization:

$$u^{r\dagger}u^s = 2E_p\,\xi^{r\dagger}\xi^s = 2E_p\,\delta_{rs} \quad \text{massless}$$

$$\bar u^r u^s = 2m\,\xi^{r\dagger}\xi^s = 2m\,\delta_{rs} \quad \text{massive}$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Negative freq. solutions: $v^s(p)$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix} \leftarrow \boxed{\text{2 indep. solutions}}$$

$$v^{r\dagger}v^s = 2E_p\,\eta^{r\dagger}\eta^s = 2E_p\,\delta_{rs} \quad \text{massless}$$

$$\bar v^r v^s = -2m\,\eta^{r\dagger}\eta^s = -2m\,\delta_{rs} \quad \text{massive}$$

$$\bar u\, v = \bar v\, u = 0 \longrightarrow \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} - \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} = 0$$

$$(p\cdot\sigma)(p\cdot\bar\sigma) = m^2$$

$$u^{+}(p)\, v(p) \neq 0$$

but $$v^{+}(\vec{p})\, v(-\vec{p}) = v^{+}(-p)\, u(p) = 0$$

Recall: QM $\quad |\varphi_n\rangle$ – orthonormal vectors

$$\langle \varphi_m | \varphi_n \rangle = \delta_{nm} \qquad \sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbb{1}$$

$$\sum_{s=1,2} u^s \bar{v}^s = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \quad \xi^{s\dagger}\sqrt{p\cdot\sigma} \right) =$$

$$= \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} \end{pmatrix} = \begin{pmatrix} m\,\mathbb{1} & p\cdot\sigma \\ p\cdot\bar{\sigma} & m\,\mathbb{1} \end{pmatrix}$$

$$\sum_s \xi^s \xi^{\dagger s} = \mathbb{1}_{2\times 2}$$

$$\sum_{s=1,2} u^s \bar{u}^s = \begin{pmatrix} m\,\mathbb{1} & p\cdot\sigma \\ p\cdot\bar{\sigma} & m\,\mathbb{1} \end{pmatrix} = m\,\mathbb{1} + p\cdot\gamma = \not{p} + m$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Feynman notation $\quad \gamma\cdot p = \gamma^\mu p_\mu = \not{p}$

$$\sum_{s=1,2} v^s \bar{v}^s = \not{p} - m$$

Dirac field bilinears

$$\mathcal{L}_{Dirac} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

$\bar{\psi}\gamma^\mu\psi$ - 4 vector $\qquad\qquad \bar{\psi}\psi$ - scalar

$\psi^\dagger \gamma^0 \gamma^\mu \psi$

Consider $\quad (\bar{\psi}\,\Gamma\,\psi)^\dagger = \bar{\psi}\,\Gamma\,\psi$ - observable

Solution: write Γ in terms of γ^μ

m – 2×2 Hermitian matrix

$\forall\, m \qquad m = a_i \sigma_i + a_0 \mathbb{1} \qquad \mathrm{Tr}\, m = 2a_0$

traceless

$$m = \begin{pmatrix} w & c \\ \bar{c} & w \end{pmatrix} - 4 \text{ parameters} \Rightarrow 4\ a_\mu$$

$N \times N$ Hermitian

$c = x + iy$

$$\begin{pmatrix} w & c & c & c \cdots \\ \bar{c} & w & & \\ \bar{c} & & \ddots & \\ \vdots & & & w \end{pmatrix} \qquad N \text{ of } w + \frac{N^2 - N}{2} \text{ of } c$$

$$\# \text{ of parameters} = N + (N^2 - N) = N^2$$

4×4 — $4^2 = 16$ parameters $\Rightarrow$ γ^μ — not enough

$$[A, B] = AB - BA$$

$2^{d/2}$ = min dim of rep

$$N = 2^{d/2}$$

\# of matrices

$\mathbb{1}$ 1

boosts/rot

γ^μ 4

$= \gamma^\mu \gamma^\nu$

$\gamma^{\mu\nu} = \frac{1}{2!}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu} \gamma^{\nu]} = -i\, \sigma^{\mu\nu}$ $\binom{4}{2} = 6$

$\gamma^{\mu\nu\rho} = \gamma^{[\mu} \gamma^\nu \gamma^{\rho]}$ $\binom{4}{3} = 4$

$\shortparallel$

$\gamma^\mu \gamma^\nu \gamma^\rho$

$d = 4$

$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu} \gamma^\nu \gamma^\rho \gamma^{\sigma]}$ $\binom{4}{4} = 1$

\# of antisymm. comb $= 2^d$ 16

$$\gamma^{123} = \frac{1}{3!}\Big[\gamma^1\gamma^2\gamma^3 - \gamma^2\gamma^1\gamma^3 - \gamma^1\gamma^3\gamma^2 + \gamma^3\gamma^1\gamma^2 +$$

$$+\gamma^2\gamma^3\gamma^1 - \gamma^3\gamma^2\gamma^1\Big] = \gamma^1\gamma^2\gamma^3$$

$$6 = 3!$$

$$\gamma^{112} = 0 \neq \gamma^1\gamma^1\gamma^2$$

$$\gamma^{0123} = -\gamma^{1023} = \gamma^{1032} \cdots$$

$$\parallel$$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5 \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$\gamma^\mu \qquad \gamma^2$

$\gamma^\mu \gamma^\nu$

$\gamma^\mu . \qquad \gamma^0 \gamma^2 \gamma^3 \qquad \Gamma_i \qquad i = 1, \ldots, 16$

$$\sum_i c_i \Gamma_i = 0$$

$$\sum_i c_i \underbrace{\mathrm{Tr}(\Gamma_i \Gamma_j)}_{= \# \delta_{ij}} = 0$$

$$c_j = 0$$

$$\Lambda_{1/2}^{-1}\, \partial^\mu\, \Lambda_{1/2} = \Lambda^\mu{}_\nu\, \partial^\nu$$

$$\psi \to \Lambda_{1/2}\, \psi$$

$$J^\mu = \bar{\psi}\, \partial^\mu\, \psi \quad - \text{vector}$$

$$\bar{\psi} \to \bar{\psi}\, \Lambda_{1/2}^{-1}$$

$$\bar{\psi}\psi \to \bar{\psi}\psi$$

$$\bar{\psi}\, \Lambda_{1/2}^{-1}\, \partial^\mu\, \Lambda_{1/2}\, \psi =$$

$$J \to \Lambda J$$

$$= \Lambda^\mu{}_\nu\, \underbrace{\bar{\psi}\, \partial^\nu\, \psi}_{J^\nu} = \Lambda^\mu{}_\nu\, J^\nu$$

$\bar{\psi}\, \gamma^{\mu\nu} \psi$ – 2nd rank tensor