Last time: Free particle solutions of the Dirac eq.

$(i\gamma^\mu \partial_\mu - m)\psi = 0$ Dirac $\Rightarrow$ KG

Therefore, $\psi(x) = u(p)\, e^{-ip\cdot x}$ where $p^2 = m^2$

$(\gamma^\mu p_\mu - m)\, u(p) = 0$

Go to the rest frame $p = (m, \vec{0})$

$$(m\gamma^0 - m)\, u(p_0) = m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0) = 0$$

$$\Rightarrow \quad u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

ξ - 2 component spinor

Boost along x^3

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

$$\frac{e^{\eta}+e^{-\eta}}{2} = \frac{E}{m} \qquad \frac{e^{\eta}-e^{-\eta}}{2} = p^3/m$$

$$e^{\eta} = \frac{E+p^3}{m}, \qquad e^{-\eta} = \frac{E-p^3}{m}$$

$$u(p) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}u(p_0) = \exp\left[-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]u(p_0)$$

$$\omega_{03} = -\omega_{30} = \eta$$

$$\omega_{03}S^{03} + \omega_{30}S^{30} = 2\omega_{03}S^{03}$$

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$u(p) = \exp\left[\left(-\frac{\eta}{2}\right)\underbrace{\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}}_{=\,F}\right] u(p_0) =$$

$$D^m = \begin{pmatrix} d_1^m & & 0\\ & \ddots & \\ 0 & & d_n^m\end{pmatrix}$$

$$e^D \neq \begin{pmatrix} e^{d_1} & & \\ & e^{d_2} & \\ & & \ddots \\ & & & e^{d_n}\end{pmatrix}$$

$$F^2 = \mathbb{1} \quad \Rightarrow \quad F^{2n} = \mathbb{1} \qquad F^{2n+1} = F$$

$$= \left[\cosh\frac{\eta}{2}\overbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}^{\mathbb{1}} - \sinh\frac{\eta}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix} =$$

$$= \begin{bmatrix} e^{\eta/2}\,\dfrac{1-\sigma^3}{2} + e^{-\eta/2}\,\dfrac{1+\sigma^3}{2} & 0 \\ 0 & e^{\eta/2}\,\dfrac{1+\sigma^3}{2} + e^{-\eta/2}\,\dfrac{1-\sigma^3}{2}\end{bmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$\begin{bmatrix} e^{\eta/2}\,\frac{1-\sigma^3}{2} + e^{-\eta/2}\,\frac{1+\sigma^3}{2} & 0 \\ 0 & e^{\eta/2}\,\frac{1+\sigma^3}{2} + e^{-\eta/2}\,\frac{1-\sigma^3}{2} \end{bmatrix} \cancel{\sqrt{m}} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$e^{\eta} = \frac{E+p^3}{m}, \qquad e^{-\eta} = \frac{E-p^3}{m} \qquad\qquad e^{\eta/2} = \frac{\sqrt{E+p^3}}{\cancel{\sqrt{m}}} \qquad e^{-\eta/2} = \frac{\sqrt{E-p^3}}{\cancel{\sqrt{m}}}$$

$$\begin{bmatrix} \left(\sqrt{E+p^3}\,\frac{1-\sigma^3}{2} + \sqrt{E-p^3}\,\frac{1+\sigma^3}{2}\right)\xi \\ \left(\sqrt{E+p^3}\,\frac{1+\sigma^3}{2} + \sqrt{E-p^3}\,\frac{1-\sigma^3}{2}\right)\xi \end{bmatrix} = u(p)$$

$$\begin{bmatrix} \left(\sqrt{E+p^3}\,\dfrac{1-\sigma^3}{2} + \sqrt{E-p^3}\,\dfrac{1+\sigma^3}{2}\right)\xi \\ \left(\sqrt{E+p^3}\,\dfrac{1+\sigma^3}{2} + \sqrt{E-p^3}\,\dfrac{1-\sigma^3}{2}\right)\xi \end{bmatrix} = u(p)$$

$$\frac{1-\sigma^3}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1+\sigma^3}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$\sigma = (1, \vec{\sigma})$$

$$\begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix} = \sqrt{p\cdot\sigma}$$

$$p = (E, 0, 0, p^3)$$

$$p\cdot\sigma = E - p^3\sigma^3 = \begin{pmatrix} E-p^3 & \\ & E+p^3 \end{pmatrix}$$

$$\sqrt{p\cdot\bar{\sigma}}$$

Thus, $$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

$$\sqrt{p\cdot\sigma} = \sqrt{E - p^3\sigma^3}$$

$$\sigma^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow[\text{large boost}]{p^3 \approx E} \sqrt{2E}\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \longrightarrow \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}$$

$$O\psi = \lambda\psi \qquad \sqrt{O}\,\psi = \sqrt{\lambda}\,\psi$$

$$E^2 = p^2c^2 + (mc^2)^2$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$|\uparrow\rangle + |\downarrow\rangle$$

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

$$\cos\theta/2\,|\uparrow\rangle + e^{-i\phi/2}\sin\frac{\theta}{2}\,|\downarrow\rangle$$

$$\hat{S}^2 = s(s+1) \qquad s = 1/2$$

$$(\vec{S}\cdot\hat{u})|\uparrow\rangle = |\uparrow\rangle$$

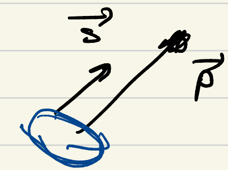
$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$S_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

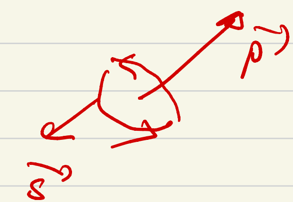
Helicity opt

$$h = \hat{p}\cdot\vec{S} = \frac{1}{2}\hat{p}_i\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$



$$\hat{p} = \frac{\vec{p}}{|\vec{p}|}$$

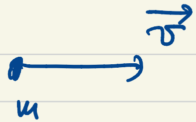
$$S_3\, u(p) = \pm\frac{1}{2}u(p)$$



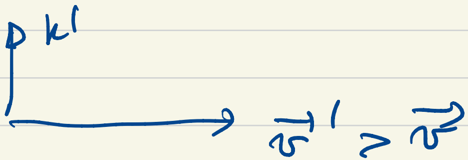
Let $\hat{p}_3 = +1$, $\hat{p}_1 = \hat{p}_2 = 0$

$h = +1/2$ – right-handed particle

$h = -1/2$ – left-handed particle



massive particle - helicity is frame-dependent



For massles particle - no such frame

Normalization: $\psi^\dagger\psi$ – not Lorentz scalar

$\psi^\dagger \to \psi\gamma^0$

$$u^\dagger u = \left(\xi^\dagger\sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix} =$$

$$E^2 - \vec p^{\,2} = m^2$$

$$= \xi^\dagger\left(p\cdot\sigma + p\cdot\bar\sigma\right)\xi = 2E_p\,\xi^\dagger\xi$$

$$(p\cdot\sigma)^\dagger = p\cdot\sigma$$

$$E_p + \cancel{\vec p\cdot\vec\sigma} + E_p - \cancel{\vec p\cdot\vec\sigma}$$

$$\bar u(p) = u^\dagger(p)\,\gamma^0 \qquad \gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\bar u\, u = \left(\xi^\dagger\sqrt{p\cdot\bar\sigma},\ \xi^\dagger\sqrt{p\cdot\sigma}\right)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix} = \xi^\dagger\, 2\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\,\xi = 2m\,\xi^\dagger\xi$$

$$(p\cdot\bar\sigma)(p\cdot\sigma) = (E_p + p_i\sigma_i)(E_p - p_j\sigma_j) = E_p^2 - p_i p_j \underbrace{\sigma_i\sigma_j}_{=\delta_{ij}}$$

$$E_p^2 - p_i p_j \delta_{ij} = m^2$$