Previously: $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}_{n\times n}$ Dirac algebra

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ - n-dim rep of Lorentz algebra

Let $\mu, \nu = 1, 2, 3, 4$ (Minkowski) and let $n = n_{min} = 4$

Weyl / chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$ σ^i - Pauli matrices

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad S^{ij} = \frac{\varepsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{\varepsilon^{ijk}}{2}\Sigma^k$$

$\Lambda_{1/2} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$ - spinor rep of Lorentz transform Λ

Ordinary 4-vectors $x \to \Lambda' x$

$$\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu$$

γ-matrices transform as 4-vectors under transformation of their spinor indices

$$\partial^\mu \partial_\mu$$

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Big\downarrow$$

$$\left[i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m\right] \Lambda_{1/2} \psi(\Lambda^{-1}x) = 0$$

$$\Lambda_{1/2}\left[i\,\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m\right]\psi =$$

$$= \Lambda_{1/2}\left[\underbrace{i\Lambda^\mu{}_\sigma \gamma^\sigma (\Lambda^{-1})^\nu{}_\mu}_{=\delta^\nu{}_\sigma} \partial_\nu - m\right]\psi =$$

$$= \Lambda_{1/2}\left[i\gamma^\nu \partial_\nu - m\right]\psi = 0 \Rightarrow \left[i\gamma^\nu \partial_\nu - m\right]\psi = 0$$

Dirac $\Rightarrow$ KG

$$a^2 - b^2 = (a-b)(a+b)$$

$$\partial^2 - m^2$$

$$0 = (-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\psi(x) =$$

$$= (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi \overset{?}{=} (\partial^2 + m^2)\psi$$

$$\parallel$$

$$\frac{1}{2}(\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + \gamma^\mu \gamma^\nu \partial_\nu \partial_\mu) = g^{\mu\nu} \partial_\nu \partial_\mu =$$

$$= \partial^\mu \partial_\mu = \partial^2$$

$m\psi$ $\qquad$ $m\psi^\dagger\psi$, but $\psi^\dagger\psi$ – not a scalar

$$\psi \to \Lambda_{1/2}\,\psi \quad \Rightarrow \quad \psi^\dagger \to \psi^\dagger \Lambda_{1/2}^\dagger$$

$$\psi^\dagger\psi \to \psi^\dagger \underbrace{\Lambda_{1/2}^\dagger \Lambda_{1/2}}_{\neq 1} \psi \neq \psi^\dagger\psi$$

Define $\bar\psi = \psi^\dagger\gamma^0 \quad \Rightarrow \quad \bar\psi \to \bar\psi\,\Lambda_{1/2}^{-1}$

$\bar\psi\psi$ – scalar, $\qquad$ $\bar\psi\gamma^\mu\psi$ – 4-vector

Let $$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

EL eqs

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \tilde{\psi})} \right) = \frac{\partial \mathcal{L}}{\partial \tilde{\psi}}$$

$\bar{\psi}$: $$(i\gamma^\mu \partial_\mu - m)\psi = 0$$ $\qquad \bar{\psi}\gamma^\mu \neq \gamma^\mu \bar{\psi}$

ψ : $$\partial_\mu(\bar{\psi} i \gamma^\mu) = -m\bar{\psi}$$

$$-i\,\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad S^{ij} = \frac{\varepsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{\varepsilon^{ijk}}{2}\Sigma^k$$

$$\psi \to e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}\psi \approx \left(1 - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)\psi$$

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \left[1 - \frac{\omega_{0i}}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \frac{i}{4}\omega_{ij}\,\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\right] \times$$

$$\times \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \to \left[1 - \frac{\omega_{0i}}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} - \frac{i}{4}\omega_{ij}\,\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\right]\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

Let $\omega_{0i} = \beta_i$; $\frac{1}{2}\varepsilon^{ijk}\omega_{ij} = \theta^k$

$$\sigma^2\psi_L^* = \left(1 + \vec{\beta}\cdot\vec{\sigma}/2 - i\,\vec{\theta}\cdot\vec{\sigma}/2\right)\sigma^2\psi_L^*$$

$$\psi_R = \left(1 + \vec{\beta}\cdot\vec{\sigma}/2 - i\,\vec{\theta}\cdot\vec{\sigma}/2\right)\psi_R$$

$\sigma^2\psi_L^*$ – transforms as ψ_R

$$\sigma^2\vec{\sigma}^* = -\vec{\sigma}\,\sigma^2$$

$$\sigma^2 \vec{\sigma}^* = -\vec{\sigma}\,\sigma^2$$

$$\sigma^2 \sigma^{1*} = -\sigma^1 \sigma^2$$

$$\sigma^2 \sigma^{2*} = -\sigma^2 \sigma^2$$

$$\sigma^2 \sigma^{3*} = -\sigma^3 \sigma^2$$

$$\sigma^{1*} = \sigma^1 \qquad \sigma^{3*} = \sigma^3$$

$$\sigma^{2*} = -\sigma^2$$

$$0 = (i\gamma^\mu \partial_\mu - m)\psi = \left[i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_0 + i \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \partial_j - \right.$$

$$\left. - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{bmatrix} -m & i(\partial_0 + \vec{\sigma} \cdot \nabla) \\ i(\partial_0 - \vec{\sigma} \cdot \nabla & -m \end{bmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Let $m = 0$

$$\begin{cases} i(\partial_0 + \vec{\sigma} \cdot \nabla)\psi_R = 0 \\ i(\partial_0 - \vec{\sigma} \cdot \nabla)\psi_L = 0 \end{cases}$$ Weyl eqs

$$(\partial_t + \partial_x) f(x,t) = 0 \qquad (\partial_t \mp \partial_x) f = 0$$

$$f = f(x - t) \qquad f(x \mp t)$$

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma)$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\begin{bmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\nabla) \\ i(\partial_0 - \vec{\sigma}\cdot\nabla & -m \end{bmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$i\,\sigma\cdot\partial\,\psi_R = 0$$

$$\begin{pmatrix} -m & i\,\sigma\cdot\partial \\ i\,\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \qquad i\,\bar{\sigma}\cdot\partial\,\psi_L = 0$$

Free particle solutions of Dirac

$$(\partial^2 + m^2)\,\psi(x) = 0 \quad \Rightarrow \quad \psi(x) = u(p)\, e^{-ip\cdot x}$$

$$p^2 = m^2$$

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

Rest frame $\quad p = (m, \vec{0})$

$$(m\gamma^0 - m)\, u(p_0) = 0$$

$$m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0) = 0$$

$$\underline{\lambda u(p_0)}$$

$$m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0) = 0 \quad \Rightarrow \quad u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$\lambda = 0$$

$$S^{ij} = \frac{\varepsilon^{ijk}}{2} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\int d^3p \, \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}} = 2\pi \int dp \, \frac{p^2}{2E_p} \, \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$\parallel$$

$$\int_0^\infty 2\pi \, p^2 \, dp \int_{-1}^{1} d(\cos\theta)^{\,x} \, \frac{1}{2E_p} e^{ipr\underbrace{\cos\theta}_{x}}$$