Previously on QFT: Lorentz group. Ordinary 4-vectors: $x \to \Lambda x$

KG scalar: $\varphi \to \varphi(\Lambda^{-1}x)$

Want Φ – n-component multiplet: $\Phi \to M\,\Phi(\Lambda^{-1}x)$

M – $n \times n$ matrix – n-dim rep of Lorentz group

Lorentz algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)$$

$J^{\mu\nu}$ – generators of Lorentz group

Group element: $M = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$

Lorentz algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)$$

rep of Lorentz group $\longrightarrow$ rep of Lorentz algebra $\longrightarrow$

$\longrightarrow$ realize comm relations in terms of matrices or diff. opts.

$J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$ – a rep (reducable)

$\left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right)$ – a 4-dim rep (ireducable)

rotations & boosts in Minkowski

$L_z \qquad \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad \dfrac{\partial}{\partial \varphi}$

A trick due to Dirac to construct an n-dim (spinor) rep

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}_{n\times n}$$ Dirac algebra

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$ - n-dim rep of Lorentz algebra

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right)$$

[Renamed $J^{\mu\nu} \to S^{\mu\nu}$]

Works in any special dim with Lorentz or Euclidean metric

Note: n - dim of rep, d - dim of space

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$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$ – n-dim rep of Lorentz algebra

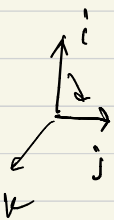
$\boxed{\begin{array}{l} n - \text{dim of rep} \\ d - \text{dim of space} \end{array}}$

Examples:

$d=1$ $\quad \gamma^1\gamma^1 + \gamma^1\gamma^1 = 2\,\mathbb{1}_{n\times n} \Rightarrow (\gamma^1)^2 = \mathbb{1}_{n\times n}$

$n_{min} = 1 \qquad \gamma^1 = \pm 1$

$d=2,3$ $\quad$ Let $\gamma^j = i\,\sigma^j \Rightarrow \{\gamma^i, \gamma^j\} = -2\,\delta^{ij}$



$$S^{ij} = -\frac{i}{4}[\sigma^i, \sigma^j] = +\frac{1}{2}\varepsilon^{ijk}\sigma^k$$

$$S^{ij} \downarrow S^k \qquad S^k = \frac{\sigma^k}{2}$$

e.g. $\quad S^3 \equiv S^{12} = \frac{1}{2}\varepsilon^{123}\sigma^3 = \frac{\sigma^3}{2}$

$n = 2$ dim rep of rotation group in $d = 2$ and 3 dim

$n_{min} = 2$

Dirac matrices for $d=4$ dim Minkowski space? $n_{min} = ?$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} \mathbb{1}_{4\times 4}$$ Dirac algebra

$1, 2, 2, 4, 4, \ldots$

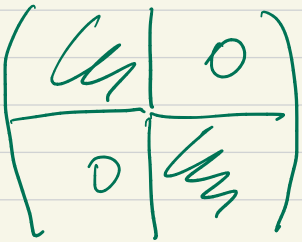
$$d = \begin{cases} 2p \\ 2p+1 \end{cases} \qquad n_{min} = 2^p$$

$$\Rightarrow \quad n_{min} = 4$$

$\sigma^1 \qquad \sigma^2$

$\sigma^1 \otimes \sigma^3 \qquad \sigma^2 \otimes \sigma^3$

$\mathbb{1} \otimes \sigma^2 \qquad \mathbb{1} \otimes \sigma^1$



$$\begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix}$$

Fact: all 4-dim rep of Dirac are unitarily equiv

$$U^\dagger \gamma^\mu U = \tilde{\gamma}^\mu$$

Weyl/chiral rep

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Boosts

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{\varepsilon^{ijk}}{2}\underbrace{\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}_{\Sigma^k} = \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

ψ – Dirac spinor

S^{0i} - non-Hermitian $\Rightarrow$ boosts are non-unitary

Lorentz group is non-compact $\Rightarrow$ no finite dim rep that are unitary

Boosts can have arbitrary θ, $\tanh\theta = v$

e.g. $e^{i\alpha J^3} \longrightarrow e^{i\alpha r} \Rightarrow |e^{i\alpha r}| = 1$

ψ - wave fn $\qquad \psi \to U\psi$

$$\left[\gamma^{\mu}, S^{\rho\sigma}\right] = \left(J^{\rho\sigma}\right)^{\mu}{}_{\nu}\, \gamma^{\nu} \qquad M(\Lambda)$$

$$\Updownarrow$$

$$\Lambda_{1/2}^{-1}\, \gamma^{\mu}\, \Lambda_{1/2} = \Lambda^{\mu}{}_{\nu}\, \gamma^{\nu}$$

$$\Lambda_{1/2} = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}$$

$$e^{\alpha A}\, B\, e^{-\alpha A} \approx B + \alpha\,[A, B]$$

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$$

$$e^{\alpha C}\, B \approx B + \alpha\, C B$$

$$[A, B] = C B \qquad e^{\alpha A}\, B\, e^{-\alpha A} = e^{\alpha C}\, B$$

$\partial^\mu \partial_\mu$ — Lorentz scalar