Two key ingredients:

- LSZ reduction
- Wick's thm

incoming $a_p^\dagger(-\infty)$ can be replaced w/ $\int d^4x\, e^{ipx}(\Box+m^2)\phi(x)$

or analogous free field eom operator

outgoing $a_p^\dagger(+\infty)$ " " $\int d^4x\, e^{-ipx}(\Box+m^2)\phi(x)$

only applies to $S-1$ or fully connected part of S-matrix

Some intuition/motivation

- replace external states w/ momentum space wavefns (will see later)

$|k\rangle$ — 1 scalars; $u_s(k)$ or $v_s(k)$ spinors; $\epsilon_\lambda(k)$ vectors

- true even in presence of arbitrary interactions limited in time!

$$\int d^4x\, e^{-ipx}(\Box+m^2)\phi = \int d^4x\, e^{-ipx}(\partial_t^2+\underbrace{p^2+m^2}_{\mathcal{E}_p^2})\phi$$

int. by parts

$$= \int d^4x \quad i\partial_t\left(e^{-ipx}(i\partial_t+\mathcal{E}_p)\phi(x)\right)$$

$$\mathcal{E}_p\, e^{-ipx}(\cancel{i\partial_t}+\mathcal{E}_p)\phi(x)$$

$$+e^{-ipx}(\partial_t^2\cancel{+i\mathcal{E}_p\partial_t})\phi(x)$$

$$= \int d^4x\, e^{-ipx}(\partial_t^2+\mathcal{E}_p^2)\phi(x)$$

$\boxed{\text{So it's a total time derivative}}$

$\downarrow$

something$(t\to+\infty)$ — something$(t\to-\infty)$

assume free theory at $t\to\pm\infty$

coupling in $\phi(x)$ mode expansion of free theory $\longrightarrow$ relate to $a^\dagger(\infty)$ and $a(-\infty)$.

So S-matrix becomes

$$\int e^{ipx_1}(\Box_1+m^2)\cdots\int e^{-ip'x'_1}(\Box'_1+m^2)\langle 0|T\,\phi(x_1)\cdots\phi(x_n)\,\phi\cdots\phi(x'_1)\cdots\phi(x'_n)|0\rangle$$

Wick's Thm: DO THIS FIRST

~~$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$ = sum of all full contractions~~

Start w/

$$D_F(x_1-x_2)\equiv\langle 0|T\,\phi(x_1)\phi(x_2)|0\rangle = \theta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle$$
$$+\theta(t_2-t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle$$

Check: $(\Box_1+m^2)D_F(x_1-x_2)$

One time derivative: $\partial_{t_1}D_F = \delta(t_1-t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle$ vanish equal time commutator!

$$+\theta(t_1-t_2)\langle 0|\partial_{t_1}\phi(x_1)\phi(x_2)|0\rangle$$
$$-\delta(t_1-t_2)\langle 0|\phi(x_2)\phi(x_1)|0\rangle$$
$$+\theta(t_2-t_1)\langle 0|\phi(x_2)\partial_{t_1}\phi(x_1)|0\rangle$$

Two time derivatives π,ϕ canonical commutator

$$\delta(t_1-t_2)\langle 0|\overbrace{[\partial_{t_1}\phi(x_1),\phi(x_2)]}^{\delta^{(3)}(x_1-x_2)}|0\rangle$$

$$\underbrace{+\theta(t_1-t_2)\langle 0|\partial_{t_1}^2\phi(x_1)\phi(x_2)|0\rangle+\theta(t_2-t_1)\langle 0|\phi(x_2)\partial_{t_1}^2\phi(x_1)|0\rangle}_{\text{combine w/ }-\nabla^2+m^2\ \to\ 0\ \text{by EOM.}}$$

So $\boxed{(\Box + m^2) D_F(x) = \delta^{(4)}(x)}$!

$\Downarrow$

$$\boxed{D_F(x) = \int d^4p \, \frac{e^{ipx}}{p^2 - m^2 + i\varepsilon}}$$

Feynman propagator!

$\Downarrow$

can also derive by closing contour for $t > 0$ above & below and $t < 0$

Can also write "contract"

$$\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = D_F(x_1 - x_2)$$

Repeating for any # of ϕ's

$$\langle 0 | T \phi(x_1) \phi(x_2) \ldots | 0 \rangle = \text{sum over all full contractions}$$

Ex

$$\langle 0 | T \phi(x_1) \ldots \phi(x_4) | 0 \rangle = D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3)(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3)$$

So general prescription for S-matrix (T-matrix actually)

$$\langle 0 | a_{k_1}(+\infty) \ldots a_{k_m}(+\infty) \, T e^{i \int d^4x \, \mathcal{H}_I} a^\dagger_{k_1}(-\infty) \ldots a^\dagger_{k_n}(-\infty) | 0 \rangle$$

- Expand in $\mathcal{H}_I$, use LSZ
- At each order, have time order corr'ln fns

$$\langle 0|\, T\, \phi(x_1)\cdots\phi(x_n)\,|0\rangle_c$$

~~next~~ exter evaluate w/ Wick contractions.

Examples:

No interactions, $2\to 2$

$$\langle 0|\, a_{k_1}(\infty)\, a_{k_2}(\infty)\, a^\dagger_{k_1'}(-\infty)\, a^\dagger_{k_2'}(-\infty)\,|0\rangle$$

|| LSZ

$$\int d^4x_1\, e^{ik_1x_1}(\Box_1+m^2)\int d^4x_2\, e^{ik_2x_2}(\Box_2+m^2)\int d^4x_1'\, e^{-ik_1'x_1'}(\Box_1'+m^2)\int d^4x_2'\, e^{-ik_2'x_2'}(\Box_2'+m^2)$$

$$\langle 0|\, T\,\phi(x_1)\,\phi(x_2)\,\phi(x_1')\,\phi(x_2')\,|0\rangle$$

|| Wick

$\to 0$ by LSZ

$$D_F(x_1-x_2)\,D_F(x_1'-x_2') + D_F(x_1-x_1')\,D_F(x_2-x_2') + D_F(x_1-x_2')\,D_F(x_2-x_1')$$

$\downarrow \Box_1^2+m^2$

$$\int d^4x_1\, e^{ik_1x_1}\,\delta^{(4)}(x_1-x_2)$$

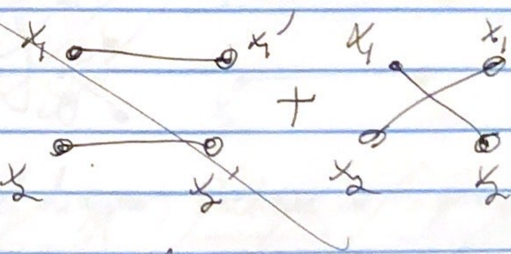
$= e^{ik_1x_2}$

$\downarrow \Box_2^2+m^2$

$$\cancel{\delta^{(4)}(k_1+k_2)}\quad \cancel{\delta^{(4)}(k_1'+k_2')}$$

~~Assume energy...~~ impossible by energy cons.

$$\delta^{(4)}(k_1-k_1')\,\delta^{(4)}(k_2-k_2') + \delta^{(4)}(k_1-k_2')\,\delta^{(4)}(k_2-k_1')$$



This is the unit matrix part of S

$S = \mathbb{1} + iT$ "no scattering"

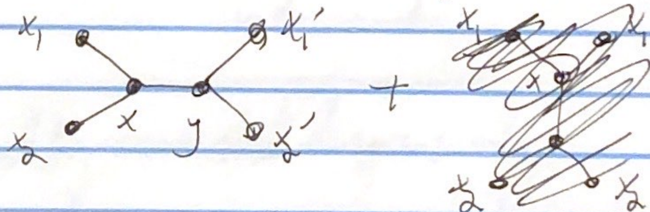
Ex: Now one power of interactions, $2 \to 2$

$g \langle 0 | a\, a \int d^4x\, \phi(x)^3\, a^\dagger a^\dagger |0\rangle = 0$ odd # of ϕ's!

Ex 2 powers of interactions, $2 \to 2$

$g^3 \langle 0 | T\, a_{k_1} a_{k_2} \int d^4x \int d^4y\, \phi(x)^3 \phi(y)^3 a^\dagger_{k_1'} a^\dagger_{k_2'} |0\rangle$

(External factors) $\langle 0 | T\, \phi(x_1)\phi(x_2)\, \phi(x)^3 \phi(y)^3\, \phi(x_1')\phi(x_2') |0\rangle$

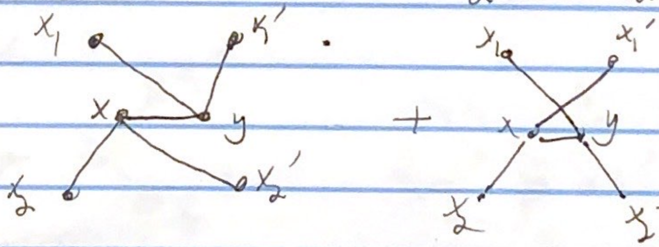


(pick $x_2 \leftrightarrow x$ WLOG)

~~Then 3~~

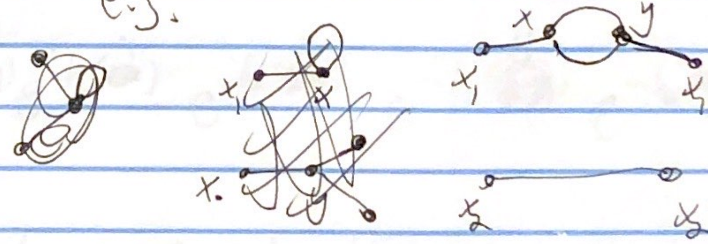
($x \leftrightarrow y$ otherwise disconnected)

(then 3 possibilities)



+ disconnected

e.g.

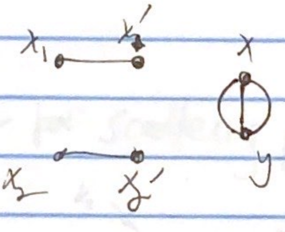


Fact: all disconnected diagrams factorize & exponentiate

~~cancel out denominator of S-matrix $\langle 0 | e^{-i\int H_I} |0\rangle$~~

Again this contributes to $\mathbb{1}$ $\longrightarrow$ only care about "fully connected"

what about



"vacuum bubbles"

$\downarrow$

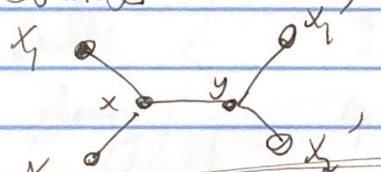
factorize

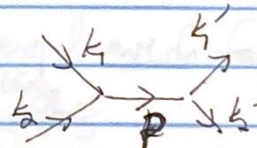
cancel out $\langle 0 | e^{-i\int \mathcal{H}_I} | 0 \rangle$

in denominator

So fully connected diagrams only

Consider





(External factors)

$$\int d^4x\, d^4y\, D_F(x-y)\, D_F(x_1-x)\, D_F(x_2-x)\, D_F(x_1'-y)\, D_F(x_2'-y)$$

$$\int d^4x\, d^4y\, d^4p\, \frac{e^{ip(x-y)}}{p^2-m^2}\, e^{ik_1(x)}\, e^{ik_2(x)}\, e^{-ik_1'(y)}\, e^{-ik_2'(y)}$$

d^4x enforces momentum cons @ x vtx $\delta^{(4)}(p+k_1+k_2)$

d^4y " " " y vtx $\delta^{(4)}(p+k_1'+k_2')$

d^4p enforces total mom. cons

Final result: $\frac{1}{(k_1+k_2)^2-m^2}\, \delta^{(4)}(k_1+k_2-k_1'-k_2')$