Check: $\partial_\mu j^\mu = 0$ (HW)

↳ eom

Noether charge $Q = \int d^3x \, j^0(x)$

$= \int d^3x \, \Psi^\dagger \Psi$

Plug in mode expansion, get

$$Q = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{s=1,2} \left(b_s^\dagger(k) b_s(k) - d_s^\dagger(k) d_s(k) \right)$$

So b & d ptcls have opposite charge!

$\Rightarrow$ ptcl & antiptcl

Back to Maxwell Lagrangian w/ sources

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \quad \text{w/ } \partial_\mu J^\mu = 0$$

required by gauge inv'a

↓

Let's use J^μ from Dirac!

introduce electric charge - coupling

$$\text{So } \mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}^2 + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi + e A^\mu \bar{\Psi} \gamma_\mu \Psi$$

Note:

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4}F^2 + i\bar{\Psi}\gamma^\mu \underbrace{(\partial_\mu - ieA_\mu)}_{\text{"covariant derivative } D_\mu\text{"}}\Psi - m\bar{\Psi}\Psi}$$

means Ψ is also inv't under local $U(1)$ gauge transf.!

$$\begin{cases} \Psi \to e^{i\varphi(x)}\Psi \\ \bar{\Psi} \to e^{-i\varphi(x)}\bar{\Psi} \\ A_\mu \to A_\mu + \partial_\mu \varphi \leftarrow \text{cancelled by } \partial_\mu(e^{i\varphi(x)}\Psi)! \end{cases}$$

$\longrightarrow$ can promote gauge symmetry to fundamental principle

Require $\mathcal{L}$ to be gauge inv't

$\Longrightarrow$ Then $\mathcal{L}_{QED}$ is unique $\mathcal{L}$ for spin 1 + spin-$\frac{1}{2}$ to ~~leading~~ order in fields

~~quadratic~~

leading

$\Longrightarrow$ interactions completely determined by gauge symmetry

$$\mathcal{L}_{int} = e\bar{\Psi}\gamma^\mu A_\mu \Psi$$

$e \to 0$ decoupled free theories

Now w/ interactions we are finally ready to calculate physical processes in QED!

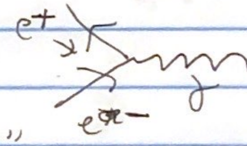
How? $\longrightarrow$ perturbation theory!

$$H_{int} = \int d^3x \; e \bar{\Psi} \gamma^\mu A_\mu \Psi$$

- e.g. 1st order $\langle f | V | i \rangle$

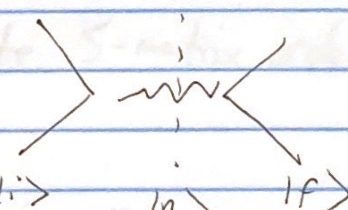
$$\langle \gamma | H_{int} | e^+ e^- \rangle$$ e^+e^- annihilation!

$\xleftarrow{\text{time}}$



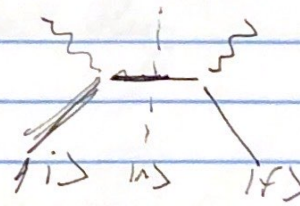
"vertex"

- 2nd order $\dfrac{\langle f | V | n \rangle \langle n | V | i \rangle}{E_i - E_n}$



e^+e^- annihilation

or



Compton scattering

etc.

⋮

very natural to visualize w/ these diagrams

$\longrightarrow$ Feynman Diagrams!

More generally, the fundamental observable of interacting QFT is the S-matrix

$$\langle f | T e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | i \rangle$$

(from TDPT, interaction picture)

"out state" time ordering time evolution operator $U_I(-\infty,\infty)$ "in state"

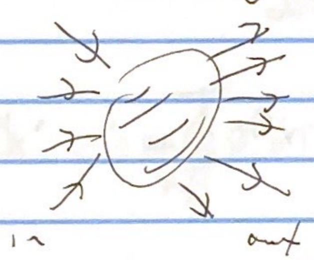
$|k_1', k_2', \ldots$
$s_1', s_2' \ldots \rangle$
or $\lambda_1', \ldots$

$(k_1, k_2 \ldots$
$s_1 \quad s_2$
$\lambda_1 \quad \lambda_2 \ldots$

$|f\rangle \sim \langle 0| a$ or b or d ...

Scattering amplitude

$|i\rangle \sim a^\dagger_{k_1} a^\dagger_{k_2} |0\rangle$ or $b^\dagger$ or $d^\dagger$ ...



Can compute S-matrix order by order in pert'n theory

e.g.

$$\langle f | T \left(-i\int_{-\infty}^{\infty} dt\, H_I(t)\right)\left(-i\int_{-\infty}^{\infty} dt'\, H_I(t')\right) | i\rangle$$

$$= \int d^4x\, d^4x' \langle f | T \mathcal{H}_I(x) \mathcal{H}_I(x') | i \rangle$$

$$= \int d^4x\, d^4x' \langle f | T\, \bar\Psi \gamma^\mu A_\mu \Psi(x)\, \bar\Psi \gamma^\nu A_\nu \Psi(x') | i\rangle$$

How to evaluate this mtx element??

- It's in a free theory!
- want to bring a's $\leftrightarrow$ $a^\dagger$'s $\quad a|0\rangle = 0$

$$\langle 0|a^\dagger = 0.$$

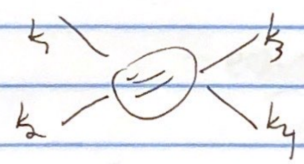
- Gotta be smarter about this...

Detour: ϕ^3 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + g\phi^3$$

$\hookrightarrow \mathcal{H}_I$

$$\int d^4x\, d^4x' \, \langle 0| a_{k_3} a_{k_4}\, T\, g\,\phi(x)^3 \quad g\,\phi(x')^3 a^\dagger_{k_1} a^\dagger_{k_2} |0\rangle$$



$$\phi \sim \int \frac{d^3k}{(\cdots)}\left(a_k e^{ikx} + a_k^\dagger e^{-ikx}\right)$$

General problem:

$$\langle 0| a_{k_1'}^{(\infty)} \cdots a_{k_n'}^{(\infty)}\, T\phi(x_1)\phi(x_2)\cdots \; a_{k_1}^{\dagger(-\infty)} \cdots a_{k_n}^{\dagger(-\infty)}|0\rangle$$

How to calculate this?