Lorentz transformations of spinor fields

Note: γ^μ doesn't transform despite have μ index

Note: Dirac spinor indices are spacetime indices, so they transform

$$\Psi_a(\Lambda x) = S_{ab}(\Lambda)\,\Psi_b(x) \quad \text{for some } S_{ab}(\Lambda)$$

How to ~~transform~~ determine $S_{ab}(\Lambda)$?

Also want: $S_{ab}(\Lambda_1)\, S_{bc}(\Lambda_2) = S_{ac}(\Lambda_1 \Lambda_2)$

So S_{ab} are a <u>representation</u> of Lorentz $SO(3,1)$

Don't want to get too much into math-fit...

Suffice to say: S_{ab} parametrized in terms of <u>generators</u>
infinitesimal transf.

Actually — so is $\Lambda^\mu{}_\nu$!

$$\Lambda = \exp\Big(\tfrac{1}{2}\underbrace{\omega_{\rho\sigma}}_{\text{params of Lorentz}} \underbrace{M^{\rho\sigma}}_{\text{antisym } 4\times 4 \text{ matrices}}\Big)$$

$\Big\downarrow$ can show...

$$S(\Lambda) = \exp\left(\tfrac{1}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right)$$

where $\boxed{S^{\rho\sigma} = \tfrac{1}{4}[\gamma^\rho, \gamma^\sigma]}$

check: $S^{ij} = -\frac{i}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ ✓.
spin rotations

Note: $S(\Lambda)^\dagger S(\Lambda) \neq \mathbb{1}$ $\rightarrow$ not unitary!

(b/c $SO(3,1)$, not $SO(4)$)

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right] \neq -S^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

True for $\underset{\text{rots}}{S^{ij}}$ but not $\underset{\text{boosts}}{S^{0i}}$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

So $\Psi^\dagger \Psi$ is not Lorentz inv't!

$\rightarrow$ Can show: $S(\Lambda)^\dagger = \gamma^0 S(\Lambda)^{-1} \gamma^0$

So $\underbrace{\Psi^\dagger \gamma^0}_{\equiv \bar{\Psi}} \Psi$ is Lorentz inv't!

$\rightarrow$ Also can show: $S(\Lambda)^{-1} \gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$

So $\bar{\Psi} \gamma^\mu \partial_\mu \Psi$ is Lorentz inv't!

$\longrightarrow$ Two terms of Dirac Lagrangian ✓.

Expect Ψ is an anticommuting field

$$\Psi(x_1, x_2, \ldots) = -\Psi(x_2, x_1, \ldots)$$

etc

Dirac Lagrangian:

$$\boxed{\mathcal{L}_{\text{Dirac}} = i\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi}$$

where $\overline{\Psi} = \Psi^\dagger\beta$

needed for Lorentz invariance

$$\boxed{\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\overline{\Psi}} = i\gamma^\mu\partial_\mu\Psi - m\Psi = 0} \checkmark$$

Ψ is a spinor field (4 indices) transforms nontrivially under Lorentz (like A_μ does)

Canonical Quant:

$$\Pi_\Psi = \frac{\partial\mathcal{L}}{\partial(\partial_0\Psi)} = i\,\overline{\Psi}\gamma^0$$

$$\mathcal{H} = \Pi_{\Psi}\partial_0\Psi - \mathcal{L} = -i\,\overline{\Psi}\,\vec{\gamma}\cdot\vec{\nabla}\Psi + m\overline{\Psi}\Psi$$

$$\cancel{\mathcal{H} = \Psi\vec{\gamma}\cdot\vec{\nabla}\Psi + m} =$$

Mode expansion

$$\boxed{\Psi(x) = \int\frac{d^3k}{(2\pi)^3 2E_k}\sum_s b_s(k)u_s(k)e^{ikx} + d_s^\dagger(k)v_s(k)e^{-ikx}}$$

$\hat{b}$ & $\hat{d}$ <u>anti</u> commuting operators

Canonical <u>anti</u> commutation relations

$$\begin{cases} \{ \Psi_a(x,t), \underbrace{\bar{\Psi}_b(y,t)}_{\text{this is } \Pi_\Psi} \} = (\gamma^0)_{ab}\, \delta^3(\vec{x}-\vec{y}) \\ \{ \Psi_a(x,t), \Psi_b(y,t) \} = 0 \end{cases}$$

~~no classical analogue!~~ an extrapolation of canonical quant...

$$\Downarrow$$

$$\{ b_s(k), b_{s'}(k') \} = \{ d_s(k), d_{s'}(k') \} = \{ b_s(k), d_{s'}^{\dagger}(k') \} = \dots = 0$$

only nonzero

$$\boxed{\begin{aligned} \{ b_s(k), b_{s'}^{\dagger}(k') \} &= (2\pi)^3 \delta^{(3)}(k-k')\, 2\epsilon_k \delta_{ss'} \\ \{ d_s(k), d_{s'}^{\dagger}(k') \} &= \quad '' \qquad '' \end{aligned}}$$

After lengthy calc:

$$\boxed{ H_{\text{Dirac}} = \sum_s \int \frac{d^3k}{(2\pi)^3 2\epsilon_k}\, \epsilon_k \left(b_s^{\dagger}(k) b_s(k) - d_s(k) d_s^{\dagger}(k) \right) }$$

Aside: $H > 0$
$\Rightarrow$ spin statistics!

(Harmonic oscillators! But obeying Fermi-Dirac statistics

$$= \dots \left(b_s^{\dagger} b_s + d_s^{\dagger} d_s \right)$$

Ground state: $\boxed{b_s(k)|0\rangle = d_s(k)|0\rangle = 0}$

Both $b^\dagger_s(k)$ & $d^\dagger_s(k)$ create positive energy states
of spin $S_z = \frac{1}{2}s$, mom. $\vec{k}$, ...
what distinguishes them?

$\longrightarrow$ charge!!

Continuous Symmetries of Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi$$

Phase rotation $\Psi \to e^{i\theta}\Psi$

$\bar{\Psi} \to e^{-i\theta}\bar{\Psi}$ $\quad U(1)$

Recall Noether's Thm: continuous symmetry $\longleftrightarrow$ conserved current

$$0 = \delta\mathcal{L}_{sym} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi = \cancel{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)}\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$$

Noether current

Consider infinitesimal transf.

$$\delta\Psi = i\varepsilon\Psi$$
$$\delta\bar{\Psi} = -i\varepsilon\bar{\Psi}$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\delta\Psi + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})}\delta\bar{\Psi}} = \bar{\Psi}\gamma^\mu\Psi$$