Friday, January 28, 2022 11:19 AM

(eigenvals ± 1) + (traceless) $\Rightarrow$ N even

$N=2 \longrightarrow \alpha_i = \sigma_i$ (maybe HW)

then β cannot exist.

$\downarrow$

$N=4$ (sol'n is not unique but they are all related by Lorentz. transf.)

$$\boxed{\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}}$$

"Dirac matrices"

let $\begin{cases} \gamma^0 = \beta \\ \gamma^i = \beta \alpha^i \end{cases} \longrightarrow \gamma^\mu = (\gamma^0, \gamma^i)$

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

Dirac matrices are Lorentz 4-vector
"Clifford Algebra"

Friday, January 28, 2022 11:26 AM

Relativistic linear Schrödinger eqn.

$$\boxed{i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \vdots \\ \psi_4\end{pmatrix} = \left(\alpha_i P_i + \beta m\right)\begin{pmatrix}\psi_1\\ \vdots \\ \psi_4\end{pmatrix}}$$ Dirac Eqn.

physical interpretation of these 4 componts?

2: spin 1/2 → spin up & down

what about the other 2? $\xrightarrow{\text{will see}}$ positron!

predicted by Dirac — confirmed! — Nobel!

Solutions to Dirac Eqn.

$$i\frac{\partial}{\partial t}\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi \qquad \psi(x) = \chi_k e^{i\vec{k}\cdot\vec{x} - iE_k t}$$

$(\text{Dirac})^2 = KG$

$$\Rightarrow \boxed{E_k^2 = k^2 + m^2}$$

eigenvalues $\pm\sqrt{k^2+m^2}$

traceless

$$\begin{pmatrix} \sqrt{k^2+m^2} & & & 0 \\ & \sqrt{k^2+m^2} & & \\ & & -\sqrt{k^2+m^2} & \\ 0 & & & -\sqrt{k^2+m^2} \end{pmatrix}$$

2 positive

2 negative energy sol'ns.

Hamiltonian unbounded from below

$E = -\sqrt{k^2+m^2}$ $\quad k \to \infty$

$E \to -\infty$.

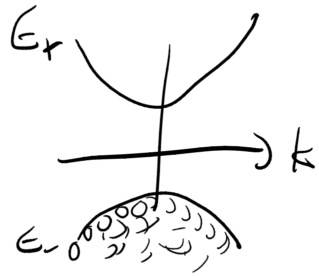
Ex: $\vec{k} = 0$

$m \neq 0$

$E\chi_0 = \beta m \chi_0$

$$= \begin{pmatrix} m & & & 0 \\ & m & & \\ & & -m & \\ 0 & & & -m \end{pmatrix}\chi_0$$

$$\chi_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$



Dirac's next idea:
maybe negative energies are filled!

Last time: Dirac Eqn:

$$i\frac{\partial}{\partial t}\psi = (\vec{\alpha}\cdot\vec{P} + \beta m)\psi \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\downarrow \qquad \vec{P} = -i\vec{\nabla} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$i\frac{\partial}{\partial t}\psi + i\vec{\alpha}\cdot\vec{\nabla}\psi = \beta m \psi$$

Define $\gamma^0 = \beta$

$\gamma^i = \beta\vec{\alpha}$

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \qquad \text{w/ } \boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

Covariant!

~~check~~ under Lorentz γ^μ transforms like 4-vector

Soln's

$$\psi(x) = \begin{cases} u_s(k)\, e^{ikx} \\ v_s(k)\, e^{-ikx} \end{cases} \qquad E_k = +\sqrt{k^2+m^2} \qquad s = 1, 2$$

$$E_k = -\sqrt{k^2+m^2}$$

spin states

$$\boxed{\begin{array}{l} (\not{k}+m)\, u_s(k) = 0 \\ (-\not{k}+m)\, v_s(k) = 0 \end{array}}$$

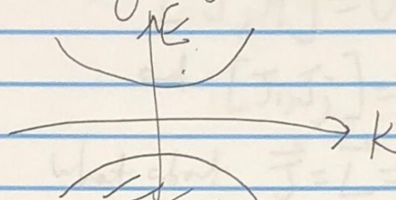
(equiv to $\vec{k} \to -\vec{k}$, $E_k \to -E_k$)

negative energies??

(Will have more to say about these sol'ns later)

Dirac's ~~bold~~ daring idea:

- electrons fermions (Pauli excl. principle)
- maybe g.s. is filled w/ ∞ sea of electrons already



Dirac Sea

QM of single ptcl $\leftarrow$

$\rightarrow$ ∞ infinite # of ptcls!

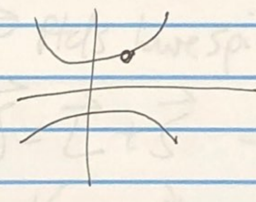
energy is now bounded from below

$E_{g.s.} = -2 \sum \sqrt{m^2+k^2}$

$= -\infty$

but OK!

- Excitations

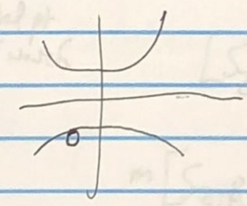
1) 

particle electron $E_k = \sqrt{k^2+m^2} > 0$

(ΔE above g.s.)

charge $= -e$

mass m

2) 

hole energy $E_k = +\sqrt{k^2+m^2} > 0$

charge $= +e$

mass m

$\downarrow$

positron antiparticle!

Cannot avoid predicting positron!

- ptcl/hole annihilation

$\rightarrow$ where does energy go?

$\rightarrow$ need photons!

$e^+e^- \to 2\gamma$ $\to$ QED.

Connection to (spin) angular momentum

Q: What is ang mom op for H? $\leftarrow m\beta + \vec{\gamma}\cdot\vec{p}$

want $[\vec{J}, H] = 0$

and $[J_i, J_j] = \epsilon_{ijk} J_k$

$[L_i, p_j] = i\epsilon_{ijk} p_k$

What about $\vec{J} = \vec{L} = \vec{r} \times \vec{p}$?

$[L_z, H] = [L_z, m\beta + \vec{\gamma}\cdot\vec{p}] = \gamma_x [L_z, p_x] + \gamma_y [L_z, p_y]$

$= i(\gamma_x p_y - \gamma_y p_x)$

So $J \neq L$

$\neq 0$!

How to fix?

$\rightarrow$ Ptcls have spin!

Try $\vec{J} = \vec{L} + \vec{S} \rightarrow$ what is $\vec{S}$

$\vec{L}$: acts on spatial pt of wavefn

$\vec{S}$: acts on internal compnts

$[S_z, H] = -[L_z, H] = -i(\gamma_x p_y - \gamma_y p_x)$

$\parallel$

$m[S_z, \beta] + [S_z, \gamma_i] p_i$

$\downarrow$

$[S_z, \beta] = [S_z, \gamma_z] = 0$

$[S_z, \gamma_x] = i\gamma_y$, $[S_z, \gamma_y] = -i\gamma_x$

etc for other components

(HW) $\rightarrow$ unique sol'n

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \equiv \frac{1}{2}\Sigma$$

$$\text{ex:} \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \quad \checkmark$$

$$\text{Conclude:} \quad \vec{J} = \vec{r}\times\vec{p} + \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$\uparrow$ 2 copies of spin $1/2$!

Mathematical Aside: Why inevitable?

Not magic $\rightarrow$ math!

"Relativity $= (\text{Spin})^2$"

Lorentz gp: $SO(3,1) \supset SO(3)$ ordinary rotations

But also $SO(3,1) \sim SO(4) \cong SU(2)_L \times SU(2)_R$

equivalent to 2 copies of spin!

ptcls classified by $SU(2)_L \times SU(2)_R$ gfm #5

$(0,0)$ scalar

$\left.\begin{matrix}(\frac{1}{2},0) \\ (0,\frac{1}{2})\end{matrix}\right\}$ spin $1/2$

$(\frac{1}{2},\frac{1}{2})$ vector

⋮

From Dirac Eqn to QED

$(i \gamma^\mu \partial_\mu - m)\psi = 0$ eom

ψ interp. as wave fn.

Just as for E&M field what if we interpret e.o.m. as field eqn ? → Dirac field $\Psi(x)$

What is $\mathcal{L}$?

$\psi(x) = \langle 0 | \Psi(x) | k \rangle$

Single ptcl wave fn. obeys same e.o.m. !

"2nd quantization"

what is

$\psi(x_1, x_2, \ldots x_N) = \langle 0 | \Psi(x_1) \cdots \Psi(x_N) | k_1, k_2, \ldots k_N \rangle$?

N ptcl state! → Fermi statistics