Unit I: Intro / Overview of QED

1. ~~Maxwell~~ E&M as a relativistic field theory (classical)

Recall: $\vec{\nabla}\cdot B = 0$

$$\vec{\nabla}\times E + \frac{\partial B}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial E}{\partial t} = 0$$

Maxwell eqns in vacuum

$\downarrow$ pot'l formalism

$(\phi, \vec{A})$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$

$$E = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

E, B inv't under $\begin{cases} \vec{A} \to \vec{A} - \vec{\nabla}\varphi \\ \phi \to \phi - \dot{\varphi} \end{cases}$ for arbitrary φ

"gauge invariance" $\to$ all physical observables inv't under gauge transf.

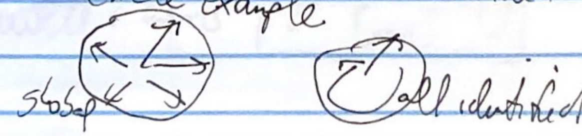
"redundancy of description"

all gauges are physically indistinguishable

"local symmetry" $\to$ A, A' related by g.t. are identified "modded out"

contrast w/ global symmetry not a redundancy of description can have different configs related by global sym are physically distinct

circle example



$\Phi = 0$ gauge

$\vec{\nabla}\cdot\vec{A} = 0 \longrightarrow \vec{\nabla}\cdot\vec{E} = 0$ automatically

(HW?) $\boxed{\nabla^2\vec{A} - \frac{\partial^2}{\partial t^2}\vec{A} = 0}$

plane wave sol'ns!

$\vec{A} \sim e^{i\vec{k}\cdot\vec{x} - \epsilon t}$

$\boxed{\epsilon^2 = k^2 \quad \text{massless relativistic dispersion relation!}}$

- Lorentz-covariant formulat'n

$A_\mu = (-\Phi, \vec{A}) \qquad A_\mu \to A_\mu - \partial_\mu \varphi$ g.t.

- gauge invariant field strength

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

check: $F_{0i} = \dot{\vec{A}} + \nabla\Phi = -\vec{E}_i$

$F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk}(\vec{\nabla}\times\vec{A})_k = \varepsilon_{ijk} B_k$

(HW)

so $F_{\mu\nu} \sim E, B$!

Check: Maxwell's eqns $\boxed{\partial^\mu F_{\mu\nu} = 0}$

Maxwell Lagrangian: (neglecting minus signs & π's & 2's)

- $S = \int d^4x\, \mathcal{L} = \int d^4x\, F_{\mu\nu}F^{\mu\nu} = \int E^2 - B^2$

Check: simplest Lorentz & Gauge inv't Lagrangian

(Lagrangians $\leftrightarrow$ relativity b/c $x \leftrightarrow t$ same footing)

- EOM S indep of $\dot{\phi} \rightarrow \phi$ not dynamical!

- Equations of motion?

Classical Lagrangian field theory: treat fields as canonical variables

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \cancel{\frac{\partial \mathcal{L}}{\partial A}} = 0$$

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

$$\Downarrow$$

$$\partial_\mu(\partial^\mu A^\nu - \cancel{\partial^\nu A^\mu}) = \partial_\mu\partial^\mu A^\nu$$

in C. gauge $\phi = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ or Lorentz gauge $\partial_\mu A^\mu = 0$

$$\Downarrow$$

free relativistic wave eqn!

$\boxed{\Box A_\nu = 0}$ $\longrightarrow$ demonstrates $\mathcal{L} \rightarrow$ Maxwell.

Mode Expansion

HW: $\frac{d^3k}{\mathcal{E}_{\vec{k}}}$ is Lorentz inv't

Work in C. gauge $A_0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$

convenience

$$\Box\vec{A} = 0 \Rightarrow \vec{A} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathcal{E}_{\vec{k}}} \left(a_{\vec{k}\lambda} \hat{e}_{\vec{k}\lambda} e^{ik\cdot x} + a^\dagger_{\vec{k}\lambda} \hat{e}^*_{\vec{k}\lambda} e^{-ik\cdot x} \right)$$

arb. cpx coeffs

- pol vectors $\hat{e}_{k\lambda} = (0, \hat{e}_{k\lambda})$ w/ $\vec{k}\cdot\hat{e}_{k\lambda} = 0$

$\vec{k}$, $\hat{e}_{k1}$, $\hat{e}_{k2}$

So 2 physical pol vectors

$\lambda = 1, 2$

- $k^2 = 0$

light is massless & travels @ speed of light

$$\boxed{\hat{e}_{k\lambda}\cdot\hat{e}^*_{k\lambda'} = \delta_{\lambda\lambda'}}$$

WLOG

Canonical Transf., $L \rightarrow H$

$\mathcal{L} = (\partial_0 A_i - \cancel{\partial_i A_0})^2 - (\partial_i A_j - \partial_j A_i) = \vec{E}^2 - \vec{B}^2$

Think of $\vec{A}(x)$ as canonical coords

Conj momenta: $\vec{\Pi} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}} = 2\dot{\vec{A}} = 2\vec{E}$

$$\mathcal{H} = \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L} = 2\dot{\vec{A}}^2 - \mathcal{L} = \vec{E}^2 + \vec{B}^2 \checkmark$$

$$= \vec{\Pi}^2 + (\vec{\nabla}\times\vec{A})^2$$

Finally, let's think of classical $\rightarrow$ quantum.

So far our treatment of E&M is classical

$\rightarrow$ We want qtm theory, where vacuum fluctuations of E&M quanta (photons) can occur, photons can be created & destroyed, etc.

How to quantize E&M?

$A_\mu(\vec{x}, t)$ usually x is operator t is label

$\downarrow$

not a good approach for relativistic thy

treat x, t on equal footing

$\downarrow$

x, t both labels!

A_μ is the operator!

~~What are its commutation relations?~~ What does it do?

$\downarrow$

IDEA: creates or destroys one quantum of the qtm field.

Schematic: $A_\mu |0\rangle \sim 1$ photon

$A_\mu A_\mu |0\rangle \sim 2$ photons

etc.

(more complicated)

What are A_μ's commutation relations? What is the Hamiltonian?
How do we calculate anything?

Canonical Quantization

Beautiful idea, applies to all Lagrangian things (not just field)
↳ "canonical"

$$\boxed{\{q, p\} = 1 \xrightarrow{\text{quantum}} [q, p] = \hbar \quad q, p \text{ now } \underline{\text{operators}}}$$

ensures classical limit!! as $\hbar \to 0$.

Apply to E&M

modulo subtleties w/ polarization & gauge

$q \to \vec{A}(x)$
$p \to \dot{\vec{A}}(x')$

so $\boxed{[\vec{A}(x), \dot{\vec{A}}(x')] = \delta^{(3)}(x - x')}$ in Quantum E&M (recall $\hbar = 1$)

not very illuminating yet...

let's use mode expansion

$$\vec{A} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 \, \epsilon_k} \left(a_{k\lambda} \hat{e}_{k\lambda} e^{ikx} + a^\dagger_{k\lambda} \hat{e}_{k\lambda} e^{-ikx} \right)$$

$$\dot{\vec{A}} = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \, i \left(a_{k\lambda} \hat{e}_{k\lambda} e^{ikx} - a^\dagger_{k\lambda} \hat{e}_{k\lambda} e^{-ikx} \right)$$

Can show: $[\vec{A}(x), \dot{\vec{A}}] = i\delta^{(3)}(x-x')$

$\Updownarrow$

$$\boxed{\begin{aligned} [a_{k\lambda}, a_{k'\lambda'}] &= [a^\dagger_{k\lambda}, a^\dagger_{k'\lambda'}] = 0 \\ [a_{k\lambda}, a^\dagger_{k'\lambda'}] &= (2\pi)^3 2E_k \, \delta^{(3)}(k-k')\delta_{\lambda\lambda'} \end{aligned}}$$

$\Downarrow$

recognize simple harmonic oscillator!!

Can show: $$\boxed{H = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 E_k} \cdot E_k a^\dagger_{k\lambda} a_{k\lambda}}$$

$H = \int d^3x \, \mathcal{H}$

+ infinite 0 pt energy

- So $a_{k\lambda}, a^\dagger_{k\lambda}$ creation/annihilation ops for photons w/ momentum $\vec{k}$ and energy $E_k = |\vec{k}|$.
- Different $\vec{k}$ & λ don't talk to eachother

$$\mathcal{H} = \sum_{k\lambda} \mathcal{H}_{k\lambda}$$

- Classical E&M is "free theory" — no interactions.
- Hilbert space $|n_{k_1\lambda_1}; n_{k_2\lambda_2}; \ldots\rangle \equiv |\{n_{k\lambda}\}\rangle$

$$E_{\{n_{k\lambda}\}} = \sum_{k\lambda} \hbar |\vec{k}| (n_{k\lambda}) \; (+ \text{ zpt energy})$$

- Vacuum $|0\rangle$ annihilated by all $a_{k\lambda}$

$a_{k\lambda}|0\rangle = 0 \;\forall k, \lambda$

note:

$\langle 0|\vec{E}|0\rangle = 0$ but $\langle 0|\vec{E}^2|0\rangle \neq 0$

as in SHO

quantum vacuum fluctuations of E&M field!

Pure Maxwell – free theory. Not very interesting as QFT... no interactions, scattering, ...

need <u>matter</u> for interactions!

Recall Maxwell + sources

$$\left\{ \begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{B} - \dot{\vec{E}} &= \vec{J} \end{aligned} \right. \longrightarrow \boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

$$\boxed{J = (\rho, \vec{J})}$$

note Maxwell eqns also imply

$\boxed{\partial_\nu J^\nu = 0}$ current conservation!

How to modify $\mathcal{L}_{maxwell}$?

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu}$$

$\downarrow$ write it all out

$A_\mu = (\Phi, A_i)$

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \frac{1}{2}(\nabla_j A_i)^2 + J_i A_i$$

$$+ \frac{1}{2}(\nabla_i \Phi)^2 - \rho\Phi$$

vary Φ: $\boxed{-\nabla^2\Phi = \rho}$

Poisson eqn

Φ still no dynamics

(still no $\dot{\Phi}$)

$$\Phi(x,t) = \int d^3y \, \frac{\rho(y,t)}{|\vec{x}-\vec{y}|}$$

$$\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \frac{1}{2}(\nabla_j A_i)^2 + J_i A_i - \underbrace{\frac{1}{2}\int d^3y \, \frac{\rho(x,t)\rho(y,t)}{|\vec{x}-\vec{y}|}}_{\mathcal{L}_{coul}}$$

$\downarrow$ same canonical transf.

(source doesn't involve $\dot{A}_i$)

same mode expansion

$$\boxed{H = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 \epsilon_k} \epsilon_k a^\dagger_{k\lambda} a_{k\lambda} - \int d^3x \, \vec{J}(x)\cdot\vec{A}(x) + H_{coul}}$$