Diverges logarithmically at large k $\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \log \Lambda$

"UV divergence" ~~[illegible]~~

Need to regularize to make any progress

$\longrightarrow$ one popular choice "dimensional regularization" (preserves gauge inv.)

Go to ~~[illegible]~~ dims $d = 4 - \varepsilon$

then $$\int \frac{d^dk}{k^4} \sim \int^{\Lambda} \frac{dk}{k^{1+\varepsilon}} \sim \Lambda^{-\varepsilon} \to 0$$ as $\Lambda \to \infty$.

regularizes UV divergence

Also: technical trick: how to evaluate integral of form

"Feynman parameters" $\frac{1}{(k^2-m_1^2)((k+p)^2-m_2^2)} \cdots$

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}$$ ~~$= \int dx\,dy\, \delta(x+y-1) \frac{1}{(xA+yB)^2}$~~

$$(1-x)(k^2-m^2) + x((p+k)^2 - m^2)$$

$$= k^2 + 2x\,p\cdot k + x p^2 - m^2$$

Shift $k \to k + xp \equiv l$

$$l^2 + \underbrace{x(1-x)p^2 - m^2}_{-\Delta}$$

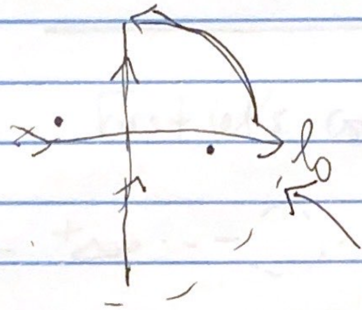
So $\ldots = \int dx \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^2}$

Last step: Wick rotate $\ell^2 = \ell_0^2 - \vec{\ell}^{\,2}$

Euclidean continuation

$\ell_0 \to i\ell_0^E$

$\vec{\ell} \to \vec{\ell}^{\,(E)}$



ℓ_0

$$\frac{1}{\ell^2 - \Delta + i\varepsilon}$$

So $\longrightarrow \int dx \int \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{(\ell_E^2+\Delta)^2}$

dim reg $4 \to 4-\varepsilon$ $\quad d = 4-\varepsilon$

Spherical coords

$$\boxed{\begin{aligned}&\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^m} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)}\frac{1}{\Delta^{m-d/2}}\\ &\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^m} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(m-\frac{d}{2}-1)}{\Gamma(m)}\frac{1}{\Delta^{m-d/2-1}}\end{aligned}}$$

$$\int \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{(\ell_E^2+\Delta)^m} = \frac{(-1)^m}{(4\pi)^2(m-1)(m-2)}\frac{1}{\Delta^{m-2}}$$

$$\int \ldots \ell^2 \ldots = \frac{(-1)^{m-1}\,2}{(4\pi)^2(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}$$

Euler–Mascheroni Const ≈ 0.5772

$$\Gamma(\tfrac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$$

So

$$= g^2\mu^{\varepsilon}\int dx \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\frac{1}{\Delta^{2-d/2}} \qquad d = 4-\varepsilon$$

$$\frac{1}{\Delta^{\varepsilon/2}} = e^{-\frac{\varepsilon}{2}\log\Delta} \approx 1 - \frac{\varepsilon}{2}\log\Delta + \ldots$$

$$\to \frac{g^2}{(4\pi)^2}\int dx\left(\frac{2}{\varepsilon} - \log\frac{\Delta}{\mu^2} - \gamma + \log 4\pi + \cdots\right)$$

$g \to g\mu^{\varepsilon/2}$

$g\phi^3$

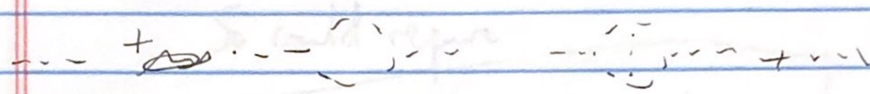
$[S] = 1$ $\quad \int d^dx$

in $d = 4-\varepsilon$ $\quad [(\partial\phi)^2] = d$

so $[\phi] = \frac{d-2}{2} = 1 - \frac{\varepsilon}{2}$ $\quad [g\phi^3] = d \to [g] = 1 + \frac{\varepsilon}{2}$

How to make sense of this infinity??

First let's consider all such diagrams



"1-PI" $\equiv \Pi(p^2)$

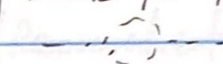
$+ \cdots$

Exact propagator $\Delta(p^2)$ $\equiv$

$$\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\Pi(p^2)\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\Pi(p^2)\frac{1}{p^2-m^2}\Pi(p^2)\frac{1}{p^2-m^2} + \cdots$$

$$= \frac{1}{p^2-m^2}\left(\frac{1}{1-\frac{\Pi(p^2)}{p^2-m^2}}\right) = \underbrace{\frac{1}{p^2-m^2-\Pi(p^2)}}_{\text{corrects propagator!}}$$

Correction: ~~Exact propagator~~ is



we have been calculating $\Pi(p^2)$ not $\Delta(p^2)$

Note: divergence Λ is just a constant → no momentum dependence (not true more generally)

suppose m^2 in Lagrangian is not true mass but just an infinite bare mass??

Physical mass given by pole in exact propagator

$$\Delta(p^2)\Big|_{p^2=m_{phys}^2} \approx \frac{1}{p^2-m_{phys}^2}$$

So could require

$$\boxed{m^2+\Pi(p^2=m_{phys}^2)=m_{phys}^2}$$

onshell "Renormalization" condition

Let $m^2=m_{phys}^2+\delta$ ← "counterterm"

So $\delta=-\Pi(p^2=m_{phys}^2)\approx\underbrace{-\Pi(p^2=m^2)}_{\sim\mathcal{O}(g^2)}+\mathcal{O}(g^4)$

$$=\frac{-g^2}{(4\pi)^2}\int dx\,\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log\frac{(1-x(1-x))m^2}{\mu^2}$$

<u>Infinite counterterm!</u>

$$=\frac{-g^2}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\log\frac{m^2}{\mu^2}+(\text{const})\right)$$

More generally, imagine including counterterms for every bare parameter in Lagrangian, e.g.

$$\frac{1}{2}Z_\phi(\partial\phi)^2-\frac{1}{2}Z_m m^2\phi^2+\frac{g}{3!}Z_g\phi^3$$

$Z_\phi=1+\delta_\phi$

$Z_m=1+\delta_m$

$Z_g=1+\delta_g$

Can cancel off all divergences w/ just 3 counterterms?