$$\begin{cases} s = (p_1+p_2)^2 = 4E^2 = E_{CM}^2 \\ t = (p_1-p_3)^2 = m_e^2 + m_\mu^2 - 2E^2 + 2|\vec{k}||\vec{p}|\cos\theta \\ u = (p_1-p_4)^2 = \text{''} \quad \text{''} - \text{''} \end{cases}$$

$\alpha = \frac{e^2}{4\pi}$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^6}\frac{|\vec{p}|}{|\vec{k}|}\left(E^4 + |\vec{k}|^2|\vec{p}|^2\cos^2\theta + E^2(m_e^2+m_\mu^2)\right)$$

$|\vec{k}| = \sqrt{E^2 - m_e^2}$

$|\vec{p}| = \sqrt{E^2 - m_\mu^2}$

$m_e \to 0$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2} + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

"phase space" "QED"

HG limit $m_\mu \to 0$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

can derive using angular momentum conservation heuristics

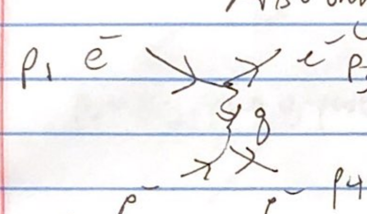
$$\sigma_{tot} = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \, \frac{d\sigma}{d\Omega} = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

total xsec. @ high energies

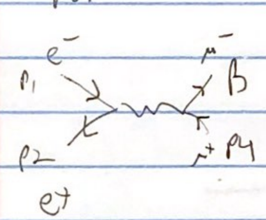
general dependence high energy by dim'l analysis

Ex 2: Rutherford Scattering $e^-\mu^- \to e^-\mu^-$

Also only 1 diagram! (could also be $e^-\mu^-\to e^-\mu^-$)



Can get from $e^+e^- \to \mu^+\mu^-$ via crossing symmetry



$\underset{\text{in } e^+}{p_2} \to -\underset{\text{out } e^-}{p_3}$ $\qquad s \to t$

$\underset{\text{out } \mu^+}{p_4} \to -\underset{\text{in } \mu^-}{p_2}$ $\qquad t \to u$

$p_3 \to p_4$ $\qquad u \to s$

$$\boxed{\frac{1}{4}\sum |M|^2 = \frac{2e^4}{t^2}\left(u^2+s^2+4t(m_e^2+m_p^2)-2(m_e^2+m_p^2)^2\right)}$$

Limits: $m_p \to \infty$ infinitely heavy proton target

holding fixed E_e, m_e

$\vec{p}_1 = (E, p, 0, 0, p)$ $\quad p_2 = (m_p, 0, 0, 0)$ $\quad p_3 = (E, p\sin\theta, 0, p\cos\theta)$

$p_4 = (m_p, 0)$

Can ignore scattering of p to leading order

$s = m_p^2 + 2Em_p + m_e^2$

$t = -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}$

$u \approx m_p^2 - 2Em_p + m_e^2$

$$\frac{1}{4}\sum |M|^2 = \frac{2e^4}{t^2}\left(8E^2m_p^2 + 4tm_p^2 + \cdots\right) = \frac{8e^4 m_p^2}{16p^4\sin^4\frac{\theta}{2}}\left(2E^2 - 2p^2(1-\cos\theta)\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}\left(\frac{1}{4}\sum |M|^2\right) =$$

skip

$p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$

$p_1 = (E_1, 0, 0, p)$ $\quad$ $p_2 = (E_2, 0, 0, -p)$

$p_4 = (E_2, -p\sin\theta, 0, -p\cos\theta)$

$E_2 = \sqrt{p^2 + m_p^2} = m_p + \frac{p^2}{2m_p} + \dots$

$s = (E_1 + E_2)^2$

$t = -2p^2(1-\cos\theta) = -4p^2 \sin^2\frac{\theta}{2}$

$u = (E_1 - E_2)^2 - 2p^2(1+\cos\theta)$

⋮

expand carefully in $m_p \to \infty$ (HW)

$\longrightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 v^2}$ $\qquad \frac{1}{4}\sum |M|^2 = \frac{e^4 m_p^2}{v^4 E^2 \sin^4\frac{\theta}{2}}\left(1 - v^2 \sin^2\frac{\theta}{2}\right)$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2} |M|^2 = \frac{e^4}{64\pi^2 v^2 p^2 \sin^4\frac{\theta}{2}}\left(1 - v^2 \sin^2\frac{\theta}{2}\right)}$$

θ

"Mott formula"

- limit $m_p \to \infty$ exists! $\to m_p$ drops out
- leading relativistic corr'n to Rutherford

$v \ll 1$

$\frac{d\sigma}{d\Omega} \sim \frac{1}{\sin^4\frac{\theta}{2}} \longrightarrow$ classic Rutherford result

can be derived in QM using Born approx

$M \sim \langle k | V_{coul}(r) | k' \rangle$

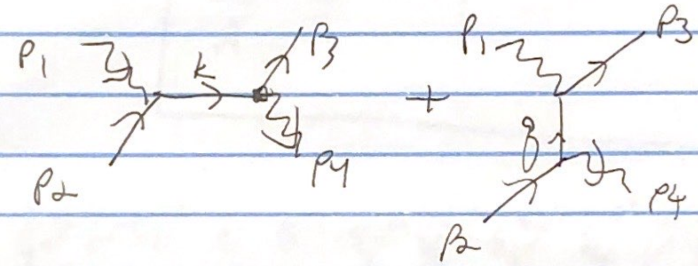
- forward singularity
  - onshell photon
- infinite σ_{tot} — V_{coul} too long ranged

~~High E limit~~

~~$E, m_p >> m_e$~~

Compton scattering

$\gamma e^- \to \gamma e^-$



$$(\mathrm{I}): e^2 \epsilon_1^\mu \epsilon_4^{*\nu} \bar{u}(p_3)\gamma_\nu \frac{\not k + m}{k^2 - m^2}\gamma_\mu u(p_2)$$

$$(\mathrm{II}): e^2 \epsilon_1^\mu \epsilon_4^{*\nu} \bar{u}(p_3)\gamma_\mu \frac{\not q + m}{q^2 - m^2}\gamma_\nu u(p_2)$$

$$\mathcal{M} = \mathrm{I} + \mathrm{II} = e^2 \epsilon_1^\mu \epsilon_4^{*\nu} \bar{u}(p_3) M_{\mu\nu} u(p_2)$$

$$\frac{1}{4}\sum_{\text{pol \& spin}} |\mathcal{M}|^2 \longrightarrow \sum_\lambda \epsilon^\mu_{p\lambda}(p)\,\epsilon^{*\mu'}_{p\lambda}(p)$$

→ recall: physical polarizations are transverse

$\epsilon^\mu \propto p^\mu$ is pure gauge

↓

Ward identity

$p^\mu \mathcal{M}_\mu = 0$ $\forall$ external photon scatt amp.

Ex

$\epsilon_1^\mu = (0,1,0,0)$

$\epsilon_2^\mu = (0,0,1,0)$

$p = (E,0,0,E)$

$\bar{p} \equiv (E,0,0,-E)$

$$\sum_\lambda \epsilon^\mu_\lambda(p)\,\epsilon^{*\nu}_\lambda(p) = -\eta^{\mu\nu} + \frac{p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu}{2E^2}$$ ← vanish by Ward id.

Can generalize to arb basis of pol vectors

$$\boxed{\sum_\lambda \varepsilon^\mu_\lambda(p)\,\varepsilon^{*\nu}_\lambda(p) \longrightarrow -g^{\mu\nu}}$$

$$\frac{1}{4}\sum|\mathcal{M}|^2 = e^4\, \varepsilon_1^\mu \varepsilon_4^{*\nu} \varepsilon_1^{*\mu'} \varepsilon_4^{\nu'}\, \bar{u}(p_3) M_{\mu\nu} u(p_2)\, \bar{u}(p_2) M_{\mu'\nu'} u(p_3)$$

$$= e^4 g^{\mu\mu'} g^{\nu\nu'}\, \mathrm{Tr}\left(M_{\mu\nu}(\not{p}_2+m) M_{\mu'\nu'}(\not{p}_3+m)\right)$$

$$= e^4\, \mathrm{Tr}\left(M_{\mu\nu}(\not{p}_2+m) M^{\mu\nu}(\not{p}_3+m)\right)$$

Real

$\downarrow$ A lot of algebra! products of 8 Dirac matrices!

$p_{ij} \equiv p_i^\mu p_{j\mu}$

$$2e^4\left(\frac{p_{24}}{p_{12}} + \frac{p_{12}}{p_{24}} + 2m^2\left(\frac{1}{p_{12}} - \frac{1}{p_{24}}\right) + m^4\left(\frac{1}{p_{12}} - \frac{1}{p_{24}}\right)^2\right)$$

(Surprisingly simple!) $\hookrightarrow$ cf "helicity spinor formalism"

Go to lab frame

~~Simplifying limits~~

~~Low energy~~ ~~cm~~ lab frame

let E of γ be ω

$p_4 = (\omega', \omega'\sin\theta, 0, \omega'\cos\theta)$

$p_1 = (\omega, 0, 0, \omega)$ $\quad p_2 = (m, 0, 0, 0)$

$p_3 = p_1 + p_2 - p_4 = (E', -p'\sin\theta', 0, p'\cos\theta')$