1 Physics 613: Problem Set 2 (*due Friday March 20*)

1.1 Spin and the Dirac Equation

1. Verify that $[L_i, P_j] = i\epsilon_{ijk} P_k$ where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum operator.

2. Verify that $[L_z, H_{Dirac}] = i(\alpha_x P_y - \alpha_y P_x)$. What are $[L_x, H_{Dirac}]$ and $[L_y, H_{Dirac}]$?

3. Verify that with $\mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \equiv \frac{1}{2}\mathbf{\Sigma}$, the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ commutes with H_{Dirac}.

1.2 Solutions to the Dirac Equation

In class we introduced the solutions to the Dirac equation $\psi(x) = u_s(k)e^{-ikx}$ and $\psi(x) = v_s(k)e^{ikx}$.

1. Verify by plugging into the Dirac equation that u_s and v_s satisfy

$$(\not{k} - m)u_s(k) = 0 \tag{1}$$

and

$$(\not{k} + m)v_s(k) = 0 \tag{2}$$

2. By considering the eigenvalues of $\not{k} - m$ and $\not{k} + m$, prove that there are exactly two independent u_s and two independent v_s solutions for every k (so $s = 1, 2$).

3. Show that the *helicity operator* $h = \frac{\mathbf{P}\cdot\mathbf{\Sigma}}{|\mathbf{P}|}$ commutes with the Dirac Hamiltonian $H_{Dirac} = \boldsymbol{\alpha} \cdot \mathbf{P} + \beta m$ and find the eigenvalues of h.

4. For momentum in the z direction (i.e. $\mathbf{k} = (0, 0, k)$), find explicitly the solutions u_s and v_s classified by eigenvalues of the helicity operator.

1.3 Dirac Hamiltonian and charge

1. Substitute the mode expansion

$$\Psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \sum_s \left(b_s(\mathbf{k}) u_s(\mathbf{k}) e^{-ikx} + d_s^\dagger(\mathbf{k}) v_s(\mathbf{k}) e^{ikx} \right) \tag{3}$$

into the Dirac Hamiltonian

$$H = \int d^3x \left(-i\bar{\Psi}\gamma^i \partial_i \Psi(x) + m\bar{\Psi}\Psi(x) \right) \tag{4}$$

and, using the canonical commutation relations for b_s and d_s, derive

$$H = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_\mathbf{k}} E_\mathbf{k} (b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) + d_s^\dagger(\mathbf{k}) d_s(\mathbf{k})) \tag{5}$$

2. Do the same for the charge operator

$$Q = \int d^3x \Psi^\dagger \Psi \tag{6}$$

and derive

$$Q = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_\mathbf{k}} (b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) - d_s^\dagger(\mathbf{k}) d_s(\mathbf{k})) \tag{7}$$

1.4 Dirac Matrix Identities

Prove the following identities involving Dirac matrices:

1. $\mathrm{Tr}(\gamma^\mu) = 0$
2. $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$
3. $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\eta^{\mu\nu}\eta^{\rho\sigma} - 4\eta^{\mu\rho}\eta^{\nu\sigma} + 4\eta^{\mu\sigma}\eta^{\nu\rho}$
4. $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$
5. $(\bar{f}\gamma^{\mu_1} \dots \gamma^{\mu_n} f')^* = \bar{f}'\gamma^{\mu_n} \dots \gamma^{\mu_1} f$ where f and f' can be any Dirac spinor (i.e. u_s or v_s).

1.5 Rutherford Scattering

In class we used crossing symmetry to transform $e^+e^- \to \mu^+\mu^-$ into $e^-\mu^- \to e^-\mu^-$ (in class we called it p instead of μ^- but it doesn't matter); the answer for the squared and summed/averaged matrix element is

$$\frac{1}{4}|\mathcal{M}|^2 = \frac{2e^4}{t^2}(u^2 + s^2 + 4t(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2) \tag{8}$$

1. Rederive this directly from the t-channel Feynman diagram for $e^-\mu^- \to e^-\mu^-$ scattering (thereby verifying explicitly the validity of crossing symmetry in this example).

2. Carefully take the $m_\mu \to \infty$ limit and derive the Mott formula:

$$\frac{d\sigma}{d\Omega}\Big|_{m_\mu\to\infty} = \frac{e^4}{64\pi^2 v^2 p^2 \sin^4\frac{\theta}{2}}(1 - v^2\sin^2\frac{\theta}{2}) \tag{9}$$

where $v = p/E$ and p and E are the 3-momentum and energy of the incoming electron respectively. [Be careful! I think the treatment in Matt Schwartz's book may not be completely correct!]