Midterm solutions (2026)

1. (a) Since $E(S, L, N)$ is homogeneous in S, L, N, it satisfies:

$$E = TS + fL + \mu N.$$

Then $dE = TdS + SdT + fdL + Ldf +$
$+ \mu dN + Nd\mu$, or

$$\underbrace{SdT + Ldf + Nd\mu = 0}_{\text{GD equation}}$$

(b) Consider

$$dE = \left(\frac{\partial E}{\partial S}\right)_L dS + \left(\frac{\partial E}{\partial L}\right)_S \overset{\swarrow\, dN = 0 \text{ here}}{dL}, \quad \text{yielding}$$

$$\left(\frac{\partial E}{\partial L}\right)_T = \underbrace{\left(\frac{\partial E}{\partial L}\right)_S}_{= f} + \underbrace{\left(\frac{\partial E}{\partial S}\right)_L}_{= T} \left(\frac{\partial S}{\partial L}\right)_T = f + T\left(\frac{\partial S}{\partial L}\right)_T.$$

Next, use $A = E - TS \Rightarrow dA = \underbrace{f}_{\left(\frac{\partial A}{\partial L}\right)_S} dL - \underbrace{S}_{-\left(\frac{\partial A}{\partial T}\right)_L} dT$

Maxwell relation: $\left(\frac{\partial f}{\partial T}\right)_L = -\left(\frac{\partial S}{\partial L}\right)_T.$

Thus, $\left(\frac{\partial E}{\partial L}\right)_T = f - \underbrace{T\left(\frac{\partial f}{\partial T}\right)_L}_{\text{"}f\text{, since } f = \kappa T} = 0.$

② (a) $\mathcal{L} = K - \mathcal{U} = \frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\sin^2\theta\,\dot{\varphi}^2 + \mu\cos\theta\, B$.

$\uparrow$
Lagrangian

Canonical momenta:

$$\begin{cases} p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}, \\ p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\sin^2\theta\,\dot{\varphi}. \end{cases}$$

Then the Hamiltonian is given by:

$$\mathcal{H} = p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - \mathcal{L} = \frac{p_\theta^2}{I} + \frac{p_\varphi^2}{I\sin^2\theta} -$$

$$- \frac{I}{2}\underbrace{\dot{\theta}^2}_{= \frac{p_\theta^2}{I^2}} - \frac{I}{2}\sin^2\theta\,\underbrace{\dot{\varphi}^2}_{= \frac{p_\varphi^2}{I^2\sin^4\theta}} - \mu\cos\theta\, B =$$

$$= \frac{p_\theta^2}{2I} + \frac{p_\varphi^2}{2I\sin^2\theta} - \mu\cos\theta\, B.$$

For a single dipole, the partition function is given by

$$Z = \underbrace{\frac{1}{h^2}}_{\text{2 DoF per dipole}} \int dp_\theta\, dp_\varphi\, d\theta\, d\varphi\; e^{-\beta\mathcal{H}} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{h^2} \int d\theta\, d\varphi\, e^{\beta\mu\cos\theta B} \underbrace{\int dp_\theta\, dp_\varphi\, e^{-\frac{\beta p_\theta^2}{2I} - \frac{\beta p_\varphi^2}{2I\sin^2\theta}}}_{\sqrt{2\pi}\sqrt{\frac{I}{\beta}} \times \sqrt{2\pi}\sqrt{\frac{I}{\beta}}\sin\theta} =$$

$$= \frac{2\pi I}{\beta h^2} \underbrace{\int d\theta\, d\varphi\, \sin\theta\, e^{\lambda\cos\theta}}_{\equiv I_1}, \text{ where } \underset{\text{dimensionless}}{\lambda} = \beta\mu B.$$

Now, $I_1 = 2\pi \int_{-1}^{1} du\, e^{\lambda u} = \frac{2\pi}{\lambda}(e^{\lambda} - e^{-\lambda}) =$

$u = \cos\theta$

$= 4\pi \frac{\sinh\lambda}{\lambda}.$

Finally, $z = \underbrace{\frac{8\pi^2 I}{\beta h^2}}_{} \frac{\sinh\lambda}{\lambda}.$

dimensionless constant C

Then $Z = z^N = C^N \left(\frac{\sinh\lambda}{\lambda}\right)^N.$

(b) Magnetization per spin is given by $m = \mu\cos\theta$.

single-spin

Thus, $m = \frac{\partial \log(z)}{\partial(\beta B)} = \mu \frac{\partial}{\partial\lambda} \log\left(\frac{\sinh(\lambda)}{\lambda}\right) =$

$$\boxed{=}\ \mu \frac{\lambda}{\sinh\lambda}\left[\frac{\cosh\lambda}{\lambda} - \frac{\sinh\lambda}{\lambda^2}\right] =$$

$$= \mu\left[\coth\lambda - \frac{1}{\lambda}\right].$$

In the $\beta \to 0$ limit, $\lambda \ll 1$ and

$$\coth\lambda \simeq \frac{1}{\lambda} + \frac{\lambda}{3} + O(\lambda^3).$$

Then $m \simeq \frac{\mu}{3}\beta\mu B = \frac{\beta\mu^2 B}{3} \underbrace{\sim B}_{\text{Curie's law}}.$

(c) $Z = \frac{8\pi^2 I}{\beta^2 h^2 \mu B}\sinh(\beta\mu B)$,

$$\log Z = \log\sinh(\beta\mu B) - 2\log\beta + \underbrace{\log\left(\frac{8\pi^2 I}{h^2\mu B}\right)}_{\text{const}(\beta)}.$$

Finally, $U = -\frac{\partial \log Z}{\partial\beta} = \frac{2}{\beta} - \mu B\coth(\beta\mu B)$.

$\beta \to 0$: $U \to \frac{2}{\beta} - \mu B\frac{1}{\beta\mu B} = \frac{1}{\beta} \to \infty$ dipoles have infinite kinetic energy
($T \to \infty$)

$\beta \to \infty$: $U \to -\mu B$, as expected: dipoles have 0 kinetic energy and are lined up with $\vec{B}$
($T \to 0$)