HW #3 solutions (2026)

1. 3.18

N spins in a magnetic field H:

$$E(n_1, n_2, \dots, n_N) = -\underset{\substack{\uparrow \\ \text{magnetic moment}}}{\mu} H \sum_{i=1}^{N} n_i \quad , \quad n_i = \pm 1 \ , \forall i$$

(a) β, H, N ensemble:

use $$Q = \sum_{n_1,\dots,n_N} e^{-\beta E(n_1,\dots,n_N)} = \prod_{i=1}^{N} \sum_{n=\pm1} e^{\beta\mu H n} =$$

$$= (\underbrace{2\cosh(\beta\mu H)}_{e^{\beta\mu H} + e^{-\beta\mu H}})^N .$$

Then $$\underbrace{\langle E\rangle}_{\text{internal energy}} = \frac{\partial \log Q}{\partial(-\beta)} = N\mu H \frac{e^{-\beta\mu H} - e^{\beta\mu H}}{e^{-\beta\mu H} + e^{\beta\mu H}} =$$

$$= -N\mu H \tanh(\beta\mu H).$$

(b) $$S = \frac{\langle E\rangle - A}{T} \overset{A = -\beta^{-1}\log Q}{=} k_B \log Q + k_B \beta \langle E\rangle =$$

$$= N k_B \log(e^{\beta\mu H} + e^{-\beta\mu H}) - N\mu H k_B \beta \tanh(\beta\mu H).$$

(c) as $T \to 0$ $(\beta \to \infty)$, we get

$\tanh(\beta \mu H) \to 1$.

Then $\langle E \rangle \xrightarrow[T \to 0]{} \underbrace{-N\mu H}$

all spins at +1, aligned with the field

$$S \xrightarrow[T \to 0]{} N k_B (\beta \mu H) - \beta \mu H (N k_B) = 0.$$

The entropy is 0 at $T=0$.

2. (a) For a single dipole,

$$\mathcal{L} = K - \mathcal{U} = \underbrace{\frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\sin^2\theta\,\dot{\varphi}^2}_{K} + \underbrace{\mu\cos\theta\, \overset{|\vec{B}|}{\overset{\shortparallel}{B}}}_{-\mathcal{U}}.$$

(b) Momenta:
$$\begin{cases} p_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta}, \\ p_\varphi = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\sin^2\theta\,\dot{\varphi}. \end{cases}$$

The Hamiltonian is given by:

$$H = K + \mathcal{U} = \frac{p_\theta^2}{2I} + \frac{p_\varphi^2}{2I\sin^2\theta} - \mu\cos\theta\, B.$$

The canonical partition function is

$Z = Z_1^N$, where Z_1 is the single-dipole partition function:

$$Z_1 = \frac{1}{h^2}\int dp_\theta\, dp_\varphi\, d\theta\, d\varphi\; e^{-\frac{H}{k_B T}} =$$

h^2 to make Z_1 dimensionless

$$= \frac{1}{h^2}\int d\theta\, d\varphi \underbrace{\int dp_\theta\, dp_\varphi\, e^{-\frac{p_\theta^2}{2Ik_BT}}\, e^{-\frac{p_\varphi^2}{2I\sin^2\theta\, k_BT}}}_{2\pi(Ik_BT)\sin\theta}\, e^{\frac{\mu\cos\theta B}{k_BT}} \quad (1)$$

$$= \frac{2\pi(I k_B T)}{h^2} \int d\theta\, d\varphi \sin\theta\, e^{\frac{\mu \cos\theta B}{k_B T}}.$$

Using $B(\lambda) = \int d\theta\, d\varphi \sin\theta\, e^{\lambda \cos\theta} = 4\pi \frac{\sinh\lambda}{\lambda}$,

we obtain: $\left(\lambda = \frac{\mu B}{k_B T}\right)$

$$Z_1 = \frac{2\pi(I k_B T)}{h^2} 4\pi \frac{\sinh\left(\frac{\mu B}{k_B T}\right)}{\frac{\mu B}{k_B T}} =$$

$$= \frac{8\pi^2 I (k_B T)^2}{\mu B h^2} \sinh\left(\frac{\mu B}{k_B T}\right).$$

(c) $m(B) = k_B T \frac{\partial \log Z_1}{\partial B} =$

$$= k_B T \frac{\partial}{\partial B}\left[\log\left(\frac{1}{B}\right) + \log\sinh\left(\frac{\mu B}{k_B T}\right) + \text{const}(B)\right] =$$

$$= k_B T \left[-\frac{B}{B^2} + \frac{\cosh\lambda}{\sinh\lambda} \frac{\mu}{k_B T}\right] =$$

$$= \frac{\mu}{\tanh\lambda} - \frac{\mu}{\lambda}.$$

③ Grand canonical partition function:

$$\Sigma = \sum_\nu e^{-\beta E_\nu + \underbrace{\beta\mu}_{\alpha} N_\nu}.$$

Then
$$\langle E\rangle = \frac{1}{\Sigma}\sum_\nu E_\nu e^{-\beta E_\nu + \underbrace{\beta\mu}_{\alpha} N_\nu} =$$
$$= -\left(\frac{\partial \log \Sigma}{\partial \beta}\right)_\alpha ,$$

$$\langle N\rangle = \frac{1}{\Sigma}\sum_\nu N_\nu e^{-\beta E_\nu + \beta\mu N_\nu} =$$
$$= \left(\frac{\partial \log \Sigma}{\partial(\beta\mu)}\right)_\beta .$$

Likewise,
$$\left(\frac{\partial^2 \log\Sigma}{\partial\beta^2}\right)_{\underbrace{\beta\mu}_{\alpha}} = -\left(\frac{\partial\langle E\rangle}{\partial\beta}\right)_{\beta\mu} =$$
$$= \langle E^2\rangle - \langle E\rangle^2 = \langle \Delta E^2\rangle,$$

$$\left(\frac{\partial^2 \log\Sigma}{\partial\alpha^2}\right)_\beta = \left(\frac{\partial\langle N\rangle}{\partial\alpha}\right)_\beta =$$
$$= \langle N^2\rangle - \langle N\rangle^2 = \langle \Delta N^2\rangle,$$

$$-\left(\frac{\partial}{\partial\beta}\left(\frac{\partial\log\Sigma}{\partial\alpha}\right)_\beta\right)_\alpha = -\left(\frac{\partial\langle N\rangle}{\partial\beta}\right)_\alpha = \langle EN\rangle - \langle E\rangle\langle N\rangle =$$
$$= \underline{\underline{\langle \Delta E\,\Delta N\rangle}}.$$