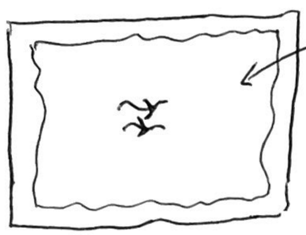
Lecture 9

Black-body radiation

EM radiation enclosed in volume V
@ temp. T: "blackbody cavity"



radiation @ equil. inside the cavity due to emission/absorption of photons by the cavity walls

Photons of freq. ω:

$$\begin{cases} E = \hbar\omega, & |\vec{k}| = \dfrac{\omega}{c} \\ \vec{p} = \hbar\vec{k} & \\ \vec{\epsilon}\cdot\vec{k} = 0, & |\vec{\epsilon}| = 1 \end{cases} \quad \Leftrightarrow \quad \vec{E}(\vec{r},t) = \vec{\epsilon}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$\uparrow$ polarization vector

$\vec{E}$ field is transverse

Use $V = L^3$, periodic BCs:

$$\vec{k} = \frac{2\pi\vec{n}}{L} \Rightarrow \underbrace{\vec{k}\cdot\vec{L}}_{\substack{\text{e.g.} \\ (L,0,0)}} \Rightarrow k_x L = 2\pi \underbrace{n_x}_{0,\pm1,\pm2,\ldots}$$

Thus, $\vec{n} = (n_x, n_y, n_z)$ where each vector component $= 0, \pm1, \pm2, \ldots$

The # of states in $(n, n+dn)$
is given by:

$$\underbrace{4\pi}_{\Omega} n^2 dn = \frac{V}{(2\pi)^3} 4\pi k^2 dk$$

Then $$E = \sum_{\vec{k},\vec{\epsilon}} \hbar \underbrace{\omega}_{= c|\vec{k}|} n_{\vec{k},\vec{\epsilon}} \qquad n_{\vec{k},\vec{\epsilon}} = 0,1,2,\ldots$$

Next, $$Q = \sum_{\{n_{\vec{k},\epsilon}\}} e^{-\beta \sum_{\vec{k},\vec{\epsilon}} \hbar\omega n_{\vec{k},\vec{\epsilon}}} =$$

$$= \prod_{\vec{k},\vec{\epsilon}} \underbrace{\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}}_{\text{geom. series}} = \prod_{\vec{k},\vec{\epsilon}} \frac{1}{1-e^{-\beta\hbar\omega}} .$$

$\omega = \omega_{\vec{k}}$

$$\log Q = -\sum_{\vec{k},\vec{\epsilon}} \log(1-e^{-\beta\hbar\omega}) = -2\sum_{\vec{k}} \log(1-e^{-\beta\hbar\omega}) .$$

Furthermore, occupation # for photons of momentum $\vec{k}$, both polarizations

$$\langle n_{\vec{k}} \rangle = \frac{\partial}{\partial(-\beta\hbar\omega)} \log Q = -\frac{2}{1-e^{-\beta\hbar\omega}}(-e^{-\beta\hbar\omega}) =$$

$$= \frac{2}{e^{\beta\hbar\omega}-1} .$$

Internal energy: $$U = \frac{\partial}{\partial(-\beta)} \log Q =$$

$$= -2\sum_{\vec{k}} \frac{1}{1-e^{-\beta\hbar\omega}}(-\hbar\omega) =$$

$$= \sum_{\vec{k}} \hbar\omega \langle n_{\vec{k}} \rangle .$$

Next, express

$$\log Q = -2 \sum_{\vec{n}} \log\Big(1 - e^{-\beta\hbar c \overbrace{\underbrace{\frac{2\pi|\vec{n}|}{V^{1/3}}}_{|\vec{k}|}}^{\omega_{\vec{k}}}}\Big).$$

Then $p = \frac{1}{\beta} \frac{\partial \log Q}{\partial V} = -\frac{2}{\beta} \sum_{\vec{n}} \frac{1}{1-e^{-\beta\hbar\omega}} \left(-e^{-\beta\hbar\omega}\right) \times$

$$\times \left(-\beta\hbar c \times 2\pi|\vec{n}|\right)\left(-\frac{1}{3V^{4/3}}\right) =$$

$$= \sum_{\vec{k}} \langle n_{\vec{k}} \rangle \frac{\hbar c \times \overbrace{\left(\frac{2\pi|\vec{n}|}{V^{1/3}}\right)}^{=|\vec{k}|}}{3V} = \frac{1}{3V} \underbrace{\sum_{\vec{k}} \hbar\omega \langle n_{\vec{k}} \rangle}_{\mathcal{U}}, \text{ s.t.}$$

$$\boxed{pV = \frac{\mathcal{U}}{3}}$$

Now, consider $(V \to \infty)$

$$\mathcal{U} = \frac{2V}{(2\pi)^3} \int_0^\infty dk \, (4\pi k^2) \frac{\hbar c k}{e^{\beta\hbar c k} - 1} =$$

$$= \frac{V\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1}.$$

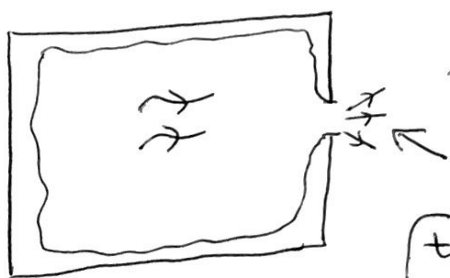
In other words,

$$\frac{\mathcal{U}}{V} = \int_0^\infty d\omega \, \underbrace{u(\omega, T)}_{\underbrace{\frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega}-1}}_{\text{energy density of photons of freq. } \omega \text{, regardless of polarization \& direction of } \vec{k}}} \Leftarrow \text{Planck's radiation law}$$

The $\int_0^\infty d\omega\, u(\omega,T)$ can be evaluated, yielding
$$\frac{U}{V} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} \sim T^4$$

$\frac{U}{V} \to \infty$ as $T \to \infty$ since the number of photons is not bounded from above.

Now, consider radiation from a small opening in the wall of the black-body cavity:



Radiation escapes with velocity c

area A

Energy radiated per second through the opening via photons of frequency ω:

$\frac{1}{2}$ of the photons point away from A

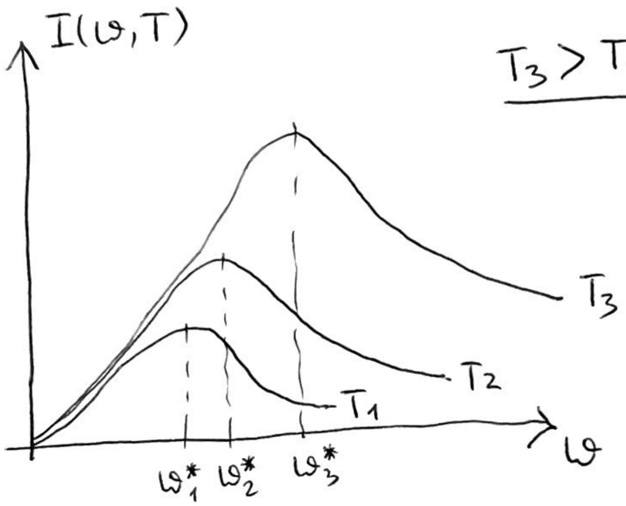
$$I(\omega,T) = \frac{c}{2} A \underbrace{\frac{u(\omega,T)}{4\pi}}_{\substack{\text{energy density per} \\ \text{unit freq., unit solid angle}}} \underbrace{\int d\Omega \cos\theta}_{2\pi} = \frac{c}{4} A\, u(\omega,T)$$

$$\int d\Omega \cos\theta = \underbrace{2}_{\substack{\text{two hemispheres,} \\ \text{pos. \& neg. } \theta}} \times 2\pi \int_0^{\frac{\pi}{2}} d\theta \times \sin\theta\cos\theta = 4\pi \int_0^{\frac{\pi}{2}} d(-\cos\theta)\cos\theta =$$

$$\overset{u=\cos\theta}{=} 4\pi \int_0^1 du\, u = 2\pi.$$

Hence $\underbrace{I(T)}_{\text{flux per unit area of the opening}} = \int_0^\infty d\omega \, \frac{I(\omega,T)}{A} = \frac{c}{4} \underbrace{\int_0^\infty d\omega \, u(\omega,T)}_{\frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}}$ =

$$= \underbrace{\frac{\pi^2 k_B^4}{60\, \hbar^3 c^2}}_{\sigma\text{, Stefan's constant}} T^4 \implies \boxed{I(T) = \sigma T^4}$$



$T_3 > T_2 > T_1$

Area under each curve is σT_i^4 $(i = 1,2,3)$

Note Consider

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \overset{u = \beta\hbar\omega}{=} \left(\frac{\hbar}{\pi^2 c^3}\right)\left(\frac{1}{\beta\hbar}\right)^4 \times$$

$$\times \underbrace{\int_0^\infty du \frac{u^3}{e^u - 1}}_{C\ \left[= \frac{\pi^4}{15}\right]} = C \frac{(k_B T)^4}{(\hbar c)^3}\frac{1}{\pi^2} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}$$ as before

Phonons in Solids

Phonons = quanta of sound waves in a macroscopic solid.

Hamiltonian $\simeq$ sum of harmonic oscillators corresponding to normal modes

quantization: $\vec{\epsilon}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \iff$ phonons, $\hbar\omega_j$

$\omega = |\vec{k}|\, c$
↑ speed of sound

polarization $\vec{\epsilon}$: 2 transverse modes + 1 longitudinal mode

Solid with N atoms $\Rightarrow$ $3N$ normal modes:

$$\omega_1, \omega_2, \ldots, \omega_{3N}$$

Each freq. ω_j can characterize multiple phonons n ($n_i = 0, 1, 2, \ldots$); the total # of phonons is not conserved.

We focus on the Debye model: normal modes are the lowest $3N$ frequencies in an elastic solid of volume V:

periodic BCs $\Rightarrow$ $\vec{k} = \dfrac{2\pi}{L}\vec{n}$, $L = V^{1/3}$

and $\vec{n} = (n_1, n_2, n_3)$, with $n_i = 0, \pm 1, \pm 2, \ldots$ ($i = 1, 2, 3$)

Then # normal modes in the $(\omega, \omega + d\omega)$ interval:

$$f(\omega)\,d\omega = \frac{3V}{(2\pi)^3} 4\pi k^2 dk = V \frac{3\omega^2}{2\pi^2 c^3} d\omega .$$

$$k = \frac{\omega}{c}$$

ω_m ← max freq.

Next, $$\int_0^{\omega_m} d\omega\, f(\omega) = 3N ,$$ or

$$\frac{3V}{2\pi^2 c^3} \underbrace{\int_0^{\omega_m} d\omega\, \omega^2}_{\frac{\omega_m^3}{3}} = 3N ,$$

$$\omega_m^3 = \frac{6\pi^2 c^3}{v} , \quad v = \frac{V}{N} .$$

Correspondingly,

$$\lambda_m = \frac{2\pi c}{\omega_m} = \left(\frac{4}{3}\pi v\right)^{1/3} \sim \underbrace{v^{1/3}}_{\text{interparticle distance}}$$

↑ min wavelength

Total energy:

$$E\{n_i\} = \sum_{i=1}^{3N} n_i \hbar \omega_i$$

$$Q = \sum_{\{n_i\}} e^{-\beta E\{n_i\}} = \prod_{i=1}^{3N} \frac{1}{1 - e^{-\beta\hbar\omega_i}} ,$$ or

$$\log Q = -\sum_{i=1}^{3N} \log\left(1 - e^{-\beta\hbar\omega_i}\right)$$

Furthermore,

$$\langle n_i \rangle = \frac{\partial}{\partial(-\beta\hbar\omega_i)} \log Q = \frac{1}{e^{\beta\hbar\omega_i}-1}.$$

Internal energy:

$$U = -\frac{\partial}{\partial\beta}\log Q = \sum_{i=1}^{3N} \hbar\omega_i \langle n_i \rangle =$$

$$= \sum_{i=1}^{3N} \frac{\hbar\omega_i}{e^{\beta\hbar\omega_i}-1}$$

In the continuous limit, $(V \to \infty)$

$$U = \frac{3V}{2\pi^2 c^3} \int_0^{\omega_m} d\omega\, \omega^2 \frac{\hbar\omega}{e^{\underbrace{\beta\hbar\omega}_{u}}-1}, \text{ or} \qquad \omega = \frac{u}{\beta\hbar}$$

$$U = \overbrace{\frac{3V}{2\pi^2 c^3} \frac{\hbar}{(\beta\hbar)^4}}^{\frac{3V}{2\pi^2 c^3}\frac{(k_B T)^4}{\hbar^3}} \int_0^{\beta\hbar\omega_m} du \frac{u^3}{e^u - 1}.$$

Note that

$$\frac{9N(k_B T)^4}{(\hbar\omega_m)^3} = \frac{9N(k_B T)^4}{\hbar^3 \frac{6\pi^2 c^3}{v}} = \frac{3V}{2}\frac{1}{\pi^2 c^3}\frac{(k_B T)^4}{\hbar^3}.$$

Thus, $$\underbrace{\frac{U}{N}}_{\text{per atom}} = \frac{9(k_B T)^4}{(\hbar\omega_m)^3} \int_0^{\beta\hbar\omega_m} du \frac{u^3}{e^u - 1}.$$

Introduce the Debye function

$$D(x) = \frac{3}{x^3}\int_0^x dt\,\frac{t^3}{e^t-1} = \begin{cases} 1-\frac{3}{8}x+\frac{1}{20}x^2+\dots & x\ll 1 \\ \frac{\pi^4}{5x^3}+\mathcal{O}(e^{-x}) & x\gg 1 \end{cases}$$

and the Debye T:

$$k_B T_D \equiv \hbar\omega_m = \hbar c\left(\frac{6\pi^2}{v}\right)^{1/3}. \Rightarrow \beta\hbar\omega_m = \frac{T_D}{T}$$

Then
$$\frac{U}{N} = 3(k_BT)\underbrace{\frac{3}{(T_D/T)^3}\int_0^{T_D/T} du\,\frac{u^3}{e^u-1}}_{D\left(\frac{T_D}{T}\right)} =$$

$$= 3(k_BT)\,D\left(\frac{T_D}{T}\right) \simeq \begin{cases} 3k_BT\left(1-\frac{3}{8}\frac{T_D}{T}+\dots\right) & T\gg T_D \\ 3k_BT\,\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3 & T\ll T_D \end{cases}$$

The specific heat:

$$\frac{C_V}{Nk_B} = 3D\left(\frac{T_D}{T}\right) + 3T\,\frac{dD\left(\frac{T_D}{T}\right)}{dT} =$$

$$= 3D\Big(\underbrace{\frac{T_D}{T}}_{=x}\Big) + 3T\,\frac{dD\left(\frac{T_D}{T}\right)}{d\left(\frac{T_D}{T}\right)\frac{dT}{d\left(\frac{T_D}{T}\right)}} =$$

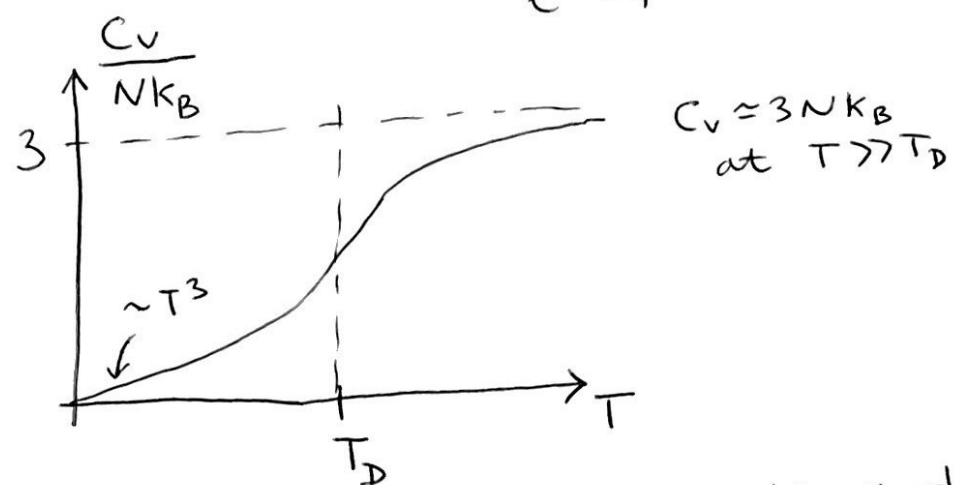
$$= 3D(x) + 3T\,\frac{d}{dx}\left[\frac{3}{x^3}\int_0^x du\,\frac{u^3}{e^u-1}\right]\times\left(-\frac{T_D}{T^2}\right) =$$

$$= 3D(x) - 3x\left[\frac{3(-3)}{x^4}\int_0^x du\,\frac{u^3}{e^u-1} + \frac{3}{\cancel{x^3}}\,\frac{\cancel{x^3}}{e^x-1}\right] =$$

$$= 3D(x) + 9D(x) - \frac{9x}{e^x-1} = 3\left(4D(x) - \frac{3x}{e^x-1}\right).$$

This leads to

$$\frac{C_V}{Nk_B} = \begin{cases} 3\left[1-\frac{1}{20}\left(\frac{T_D}{T}\right)^2+\ldots\right] & T \gg T_D \quad (x \ll 1) \\ \underbrace{\frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3}_{\text{here, } \frac{x}{e^x-1} \to 0 \text{ exponentially}} & T \ll T_D \quad (x \gg 1) \end{cases}$$



For many solids, $T_D \simeq 200\,K$ and $C_V \simeq 3Nk_B$ @ $T \simeq 300\,K$