Lecture 13

Theory of white dwarf stars

Empirical observation:

Idea: white dwarf (WD) star $\simeq$ degenerate Fermi gas

white dwarfs are at the end of stellar evolution $\Rightarrow$ H supply used up $\Rightarrow$

$\Rightarrow$ composed mainly of He.

No H supply $\Rightarrow$ low brightness
[brightness still $\neq 0$ due to gravitational energy released during a slow contraction of the star]

Typical WD star: $\rho \simeq 10^7 \, g/cm^3 \simeq 10^7 \rho_{sun}$

$M \simeq 10^{33} g \simeq M_{sun}$

$10^3 eV$ of thermal energy $\rightarrow$ $T \simeq 10^7 K \simeq T_{sun}$

He is completely ionized $\Rightarrow$ the star is a gas of He nuclei and $\bar{e}$.

The gas of $\bar{e}$ $\simeq$ an ideal Fermi gas with $\rho \simeq 10^3 \frac{\bar{e}}{cm^3}$:

$$\epsilon_F \simeq \frac{\hbar^2}{2m} \frac{1}{v^{2/3}} \simeq 20\,MeV$$

$$\hookrightarrow T_F \simeq 10^{11}\,K \gg T$$

Thus, the Fermi gas is highly degenerate $\Rightarrow$ behaves as if $T \simeq 0$ (!)

The Fermi pressure of the $\bar{e}$ gas is counteracted by the gravitational attraction, mostly due to the He nuclei. Other effects: radiation, kinetic motion of $\bar{e}$ & He nuclei may be neglected.

So, consider N relativistic $\bar{e}$'s in the ground state + $\underbrace{\frac{N}{2} \text{ stationary He nuclei}}_{\text{provide gravitational attraction}}$.

Single $\bar{e}$ state: $\vec{p}$ + $s = \pm\frac{1}{2}$.

$$\epsilon_{\vec{p}} = \sqrt{(pc)^2 + (m_e c^2)^2} \leftarrow \text{indep. of } s$$

Then $\underbrace{E_0}_{\substack{\text{ground-state} \\ \text{E of the Fermi gas}}} = 2 \sum_{p<p_F} \sqrt{(pc)^2 + (m_e c^2)^2} \; = $

$$= \frac{2V}{h^3} \int_0^{p_F} dp \, (4\pi p^2) \sqrt{(pc)^2 + (m_e c^2)^2} \; \Leftrightarrow$$

The Fermi momentum is given by

$$\frac{V}{h^3}\left(\frac{4\pi}{3} p_F^3\right) = \frac{N}{\underset{2s+1}{2}} \quad , \text{ or } \quad p_F = \hbar \left(\frac{3\pi^2}{v}\right)^{1/3} .$$

$$\Leftrightarrow N \frac{m_e^4 c^5}{\pi^2 \hbar^3} v f(x_F) \quad , \text{ where}$$

$x = \frac{p}{m_e c}$

$$f(x_F) = \int_0^{x_F} dx \, x^2 \sqrt{1+x^2} =$$

$$= \begin{cases} \frac{x_F^3}{3}\left(1 + \frac{3}{10} x_F^2 + \dots\right) & x_F \ll 1 \\ \frac{x_F^4}{4}\left(1 + \frac{1}{x_F^2} + \dots\right) & x_F \gg 1 \end{cases}$$

$$x_F \equiv \frac{p_F}{m_e c} = \frac{\hbar}{m_e c}\left(\frac{3\pi^2}{v}\right)^{1/3}$$

Next, $\underset{\substack{\uparrow \\ \text{total} \\ \text{mass of the star}}}{M} = (m_e + 2 \overset{\text{proton mass}}{m_p}) N \simeq 2 m_p N$,

$$\underset{\text{star radius}}{R} = \left(\frac{3V}{4\pi}\right)^{1/3} .$$

Then $\quad v = \dfrac{V}{N} = \dfrac{4\pi}{3} R^3 \underbrace{\dfrac{1}{N}}_{\approx \frac{2m_p}{M}} = \dfrac{8\pi}{3} \dfrac{m_p R^3}{M}.$

Next, $\quad x_F = \dfrac{\hbar}{m_e c} \dfrac{1}{R} \left(\dfrac{3}{8\pi} \dfrac{M}{m_p} 3\pi^2 \right)^{1/3} =$

$$= \frac{\hbar}{m_e c} \frac{1}{R} \left(\frac{9\pi}{8} \frac{M}{m_p} \right)^{1/3} \equiv \frac{\bar{M}^{1/3}}{\bar{R}}, \text{ where}$$

$$\begin{cases} \bar{M} = \dfrac{9\pi}{8} \dfrac{M}{m_p}, \\ \bar{R} = \dfrac{R}{(\hbar / m_e c)} \end{cases} \Leftarrow \text{dimensionless } M \;\&\; R$$

Finally, the Fermi pressure is

$$p_0 = - \frac{\partial E_0}{\partial V} = \frac{m_e^4 c^5}{\pi^2 \hbar^3} \Big[f(x_F) + \underbrace{\frac{\partial f(x_F)}{\partial x_F} \frac{\partial x_F}{\partial v} v}_{x_F^2 \sqrt{1+x_F^2}\, v \frac{\hbar}{m_e c} (3\pi^2)^{1/3} \times \left(-\frac{1}{3}\right) \frac{1}{v^{4/3}} = -\frac{1}{3} x_F^3 \sqrt{1+x_F^2}} \Big] =$$

$$= \frac{m_e^4 c^5}{\pi^2 \hbar^3} \left[\frac{1}{3} x_F^3 \sqrt{1+x_F^2} - f(x_F) \right].$$

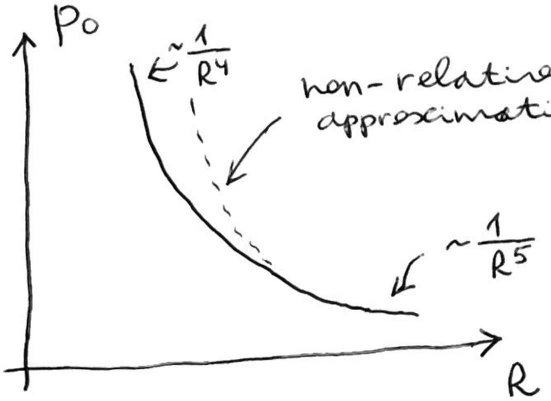
Non-relativistic limit: $x_F \ll 1$

$$p_0 \simeq \left(\frac{m_e^4 c^5}{15 \pi^2 \hbar^3} \right) x_F^5 = \frac{4}{5} K \frac{\bar{M}^{5/3}}{\bar{R}^5},$$ where

$$K = \frac{m_e c^2}{12\pi^2}\left(\frac{m_e c}{\hbar}\right)^3$$

(extreme)
<u>Relativistic</u> <u>limit</u>: $x_F \gg 1$

$$p_0 \simeq \left(\frac{m_e^4 c^5}{12\pi^2\hbar^3}\right)[x_F^4 - x_F^2] = K\left[\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2}\right]$$



$R \sim \frac{1}{x_F}$

Finally, consider

$$W = -\int_{+\infty}^{R} dr\,(4\pi r^2)\,p_0$$

thermodynamic work to 'compress' the star from $R=\infty$ to its actual radius

gravitational self-energy:

grav'l constant

$$-\alpha\frac{GM^2}{R}$$

dimensionless prefactor

at equilibrium, $\int_\infty^R p_0\,4\pi r^2\,dr = -\frac{\alpha GM^2}{R}$

$\frac{\partial}{\partial R}[\ \dots\ ]$, yielding

$$p_0 \, 4\pi R^2 = \frac{\alpha G M^2}{R^2} \Rightarrow p_0 = \frac{\alpha}{4\pi} \frac{G M^2}{R^4} =$$

$$= \frac{\alpha G}{4\pi} \left(\frac{8 m_p}{9\pi}\right)^2 \left(\frac{m_e c}{\hbar}\right)^4 \frac{\bar{M}^2}{\bar{R}^4}$$

(a) Now, assume $x_F \ll 1$ (low-density $\bar{e}$ gas):

$$p_0 \simeq \frac{4}{5} K \frac{\bar{M}^{5/3}}{\bar{R}^5} = \underbrace{\frac{\alpha G}{4\pi} \left(\frac{8 m_p}{9\pi}\right)^2 \left(\frac{m_e c}{\hbar}\right)^4}_{\text{"}K'\text{"}} \frac{\bar{M}^2}{\bar{R}^4}, \text{ or}$$

$$\underbrace{\bar{M}^{1/3} \bar{R} = \frac{4}{5} \frac{K}{K'}}_{\text{valid for small } M, \text{ large } R}$$

(b) assume $x_F \gg 1$ (high-density $\bar{e}$ gas, relativistic effects important):

$$p_0 \simeq K \left[\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2} \right] = K' \frac{\bar{M}^2}{\bar{R}^4}, \text{ or}$$

$$\bar{R}^2 \bar{M}^{2/3} = \bar{M}^{4/3} - \frac{K'}{K} \bar{M}^2,$$

$$\bar{R}^2 = \bar{M}^{2/3} - \frac{K'}{K} \bar{M}^{4/3},$$

$$(*) \quad \bar{R} = \bar{M}^{1/3} \sqrt{1 - \left(\frac{K'}{K}\right) \bar{M}^{2/3}} = \bar{M}^{1/3} \sqrt{1 - \left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}}.$$

$$\bar{M}_0 \equiv \left(\frac{K}{K'}\right)^{3/2} \Rightarrow \frac{K'}{K} = \frac{1}{\bar{M}_0^{2/3}}$$

valid for large M, small R

note that $M_0 = \frac{8}{9\pi} m_p \overline{M}_0 \simeq 10^{33}\,g \simeq M_{sun}$

$$\overline{M}_0 = \left(\frac{27\pi}{64\alpha}\right)^{3/2} \left(\frac{\hbar c}{G m_p^2}\right)^{3/2}$$ [need to know α]

Thus, it is impossible to have $\overline{M} > \overline{M}_0$ (or $M > M_0$) for any WD star $\Rightarrow$ otherwise Eq. (*) yields an imaginary radius.

Physically, the Fermi pressure is overpowered by the gravitational collapse.

Calculations of α yield $\underbrace{M_0 = 1.4\,M_{sun}}_{\text{the Chandrasekhar limit}}$

Thus, all WD stars should have $M < 1.4\,M_{sun}$ $\Leftarrow$ so far, confirmed by observations