Lecture 12

Ideal Fermi gas

Recall that $\frac{\lambda^3}{v} = f_{3/2}(z)$, where

$$v = \frac{V}{N}, \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}.$$

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^2}{1+z^{-1}e^{x^2}}.$$

$\underline{z \ll 1}$: $f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \ldots$

$\underline{z \gg 1}$: define $\nu = \log z = \beta\mu$, then

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^2}{e^{x^2-\nu}+1} = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \, \frac{\sqrt{y}}{e^{y-\nu}+1} \ominus$$

$$y = x^2 \Rightarrow dy = 2x\,dx, \quad dx = \frac{dy}{2\sqrt{y}}$$

$$\ominus \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \, \frac{y^{3/2} e^{y-\nu}}{(e^{y-\nu}+1)^2} \ominus$$

$\int$ by parts

$$\ominus \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \, \frac{e^{y-\nu}}{(e^{y-\nu}+1)^2} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2}(y-\nu) + \ldots\right]$$

$$= \frac{4}{3\sqrt{\pi}} \int_{-\nu}^\infty dt \, \frac{e^t}{(e^t+1)^2} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2} t + \ldots\right].$$

$t = y - \nu$

can extend to $-\infty$, $O(e^{-\nu})$ contribution, ν large

So, $f_{3/2}(z) = \dfrac{4}{3\sqrt{\pi}}\left[I_0 \nu^{3/2} + \dfrac{3}{2} I_1 \nu^{1/2} + \dots \right]$, where

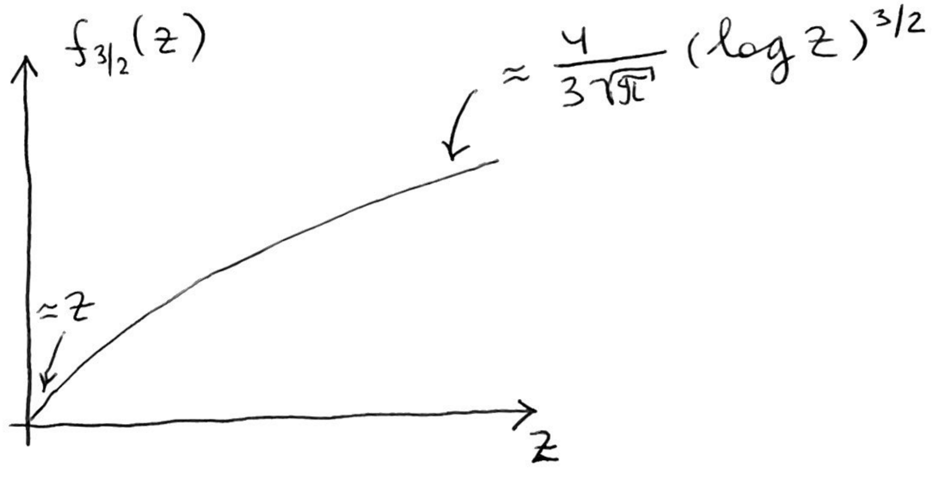
$$I_n = \int_{-\infty}^{\infty} dt \, \frac{t^n e^t}{(e^t+1)^2} .$$

$$\begin{cases} I_n = 0 , & \text{odd } n \\ I_n = (n-1)!\,(2n)\,(1-2^{1-n})\,\zeta(n) , & \text{even } n>0 \\ \quad n>0 & \\ I_0 = 1 , & n=0 \end{cases}$$

In particular, $I_2 = 1! \times 4 \times (1-\frac{1}{2}) \underbrace{\zeta(2)}_{= \frac{\pi^2}{6}} =$

$= \dfrac{\pi^2}{3}$.

Then $f_{3/2}(z) = \dfrac{4}{3\sqrt{\pi}}\left[(\log z)^{3/2} + \dfrac{\pi^2}{8} (\log z)^{-1/2} + \dots \right]$



$z \nearrow$ as $\dfrac{\lambda^3}{v} \left(= f_{3/2}(z)\right) \nearrow$

Now, consider $\underbrace{\frac{\lambda^3}{v} \ll 1}$: $v^{1/3} \gg \lambda$, quantum effects small

In this case, $\frac{\lambda^3}{v} \simeq z - \frac{z^2}{2^{3/2}}$. $\Rightarrow z = \frac{\lambda^3}{v} + \mathcal{O}\left(\left(\frac{\lambda^3}{v}\right)^2\right)$

As $T \to \infty$, $\lambda \to 0 \Rightarrow z = e^{\mu\beta} \sim p$, or
$(\beta \to 0)$

$$\underbrace{\mu\beta = \log p}_{\text{ideal gas}} + \text{const}$$

$$\langle n_{\vec{p}} \rangle = \frac{z e^{-\beta\epsilon_{\vec{p}}}}{1 - z e^{-\beta\epsilon_{\vec{p}}}} \underset{z \ll 1}{\simeq} \underbrace{\frac{\lambda^3}{v} e^{-\beta\epsilon_{\vec{p}}}}_{\text{Maxwell-Boltzmann}}.$$

Equation of state:

$$\frac{pv}{k_B T} = \frac{v}{\lambda^3} f_{5/2}(z) \underset{z \ll 1}{\simeq} \frac{v}{\lambda^3}\left[z - \frac{z^2}{2^{5/2}} + \dots\right] =$$

$$\overset{z = \frac{\lambda^3}{v}}{=} 1 - \frac{1}{2^{5/2}} \frac{\lambda^3}{v} + \dots$$

This is the EoS for ideal gas plus a small quantum correction.

Next, consider $\underbrace{\frac{\lambda^3}{v} \gg 1}$: $v^{1/3} \ll \lambda$, quantum effects significant

In this case, $\frac{1}{v}\underbrace{\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}}_{\lambda^3} \simeq \frac{4}{3\sqrt{\pi}}(\log z)^{3/2}$, or

$$(+) \quad \underbrace{\log z}_{\substack{\text{III} \\ \beta\epsilon_F}} \simeq \left(\frac{2\pi\hbar^2}{mk_BT}\right)\left(\frac{3\sqrt{\pi}}{4v}\right)^{2/3},$$

$$\frac{2^{3/2}3}{4} = \frac{2^{3/2}\,3\times 2^{3/2}}{2^{3/2}\times 4} = \underline{\underline{\frac{6}{2^{3/2}}}}.$$

$$(*) \quad \epsilon_F = \underbrace{\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{2/3}}.$$

Fermi energy (chemical potential @ $T=0$)

$z = e^{\beta\epsilon_F}$ as $T\to 0$

Recall that $\langle n_{\vec{p}}\rangle = \frac{1}{1+z^{-1}e^{\beta\epsilon_{\vec{p}}}} =$

$$\overset{\text{low}T}{=} \frac{1}{1+e^{\beta(\epsilon_{\vec{p}}-\epsilon_F)}}.$$

Clearly, as $T\to 0$, $\langle n_{\vec{p}}\rangle = \begin{cases} 1, & \epsilon_{\vec{p}} < \epsilon_F \\ 0, & \epsilon_{\vec{p}} > \epsilon_F \end{cases}$

In the ground state, particles fill up all energy levels up to ϵ_F. In momentum space, the particles fill a sphere of some radius p_F $\Rightarrow$ the surface of the sphere is the Fermi surface.

$$\epsilon_F = \frac{p_F^2}{2m}$$

Now, assume particles of spin S, then

$$\underbrace{(2S+1)}_{\text{each energy level contains } 2S+1 \text{ particles}} \sum_{\vec{p}} \langle n_{\vec{p}} \rangle \underset{T\to 0}{=} N$$

Then $(2S+1)\dfrac{V}{(2\pi\hbar)^3}\displaystyle\int_0^{p_F} dp\, p^2\, 4\pi = N$, or

$$\frac{N}{V} = \frac{2S+1}{(2\pi\hbar)^3}\frac{4\pi}{3}p_F^3 \,, \quad \text{yielding}$$

$$\boxed{\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{(2S+1)v}\right)^{2/3}}$$

consistent with Eq. (*) if $S=0$

Chemical potential:

$$\mu = k_B T \log z .$$

Recall that $\dfrac{\lambda^3}{v} = f_{3/2}(z) \simeq \dfrac{4}{3\sqrt{\pi}}\left[(\log z)^{3/2} + \dfrac{\pi^2}{8}(\log z)^{-1/2}\right] = \dfrac{4}{3\sqrt{\pi}}(\log z)^{3/2}\left[1 + \underbrace{\dfrac{\pi^2}{8}(\log z)^{-2}}_{\text{small prm}}\right].$

Then $(\log z)^{3/2} \simeq \dfrac{\lambda^3}{v}\dfrac{3\sqrt{\pi}}{4}\left(1 - \dfrac{\pi^2}{8}(\underbrace{\log z}_{= \beta\epsilon_F \text{ to this order}})^{-2}\right),$

$$\log z \simeq \underbrace{\left(\frac{\lambda^3}{v}\frac{3\sqrt{\pi}}{4}\right)^{2/3}}_{= \beta\epsilon_F \text{ from Eq. (+) (by definition)}}\left[1 - \underbrace{\frac{\pi^2}{8}\frac{2}{3}}_{\pi^2/12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right].$$

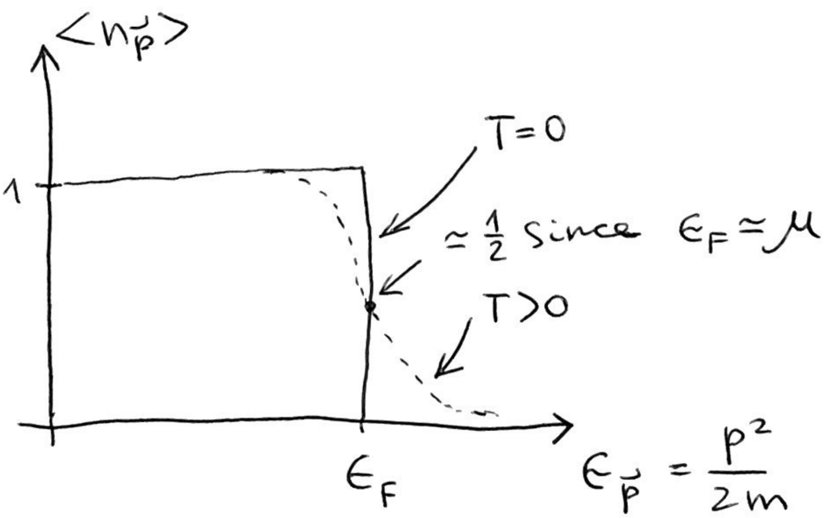
Finally,

$$\mu = k_B T \log z \approx \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] \qquad (**)$$

$\uparrow$ expansion in $\frac{k_B T}{\epsilon_F}$

Define the Fermi temperature T_F by $k_B T_F \equiv \epsilon_F$, then the expansion is valid for $\underline{T \ll T_F}$.

Next, $\langle n_{\vec{p}} \rangle = \dfrac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1}$, where μ is given by Eq. (**) in the $T \ll T_F$ regime.



Now, the internal energy

$$\mathcal{U} = \sum_{\vec{p}} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle = \frac{V}{h^3} \frac{4\pi}{2m} \int_0^\infty dp\, p^4 \langle n_{\vec{p}} \rangle =$$

$$\underset{\substack{\uparrow \\ \text{by parts}}}{=} \frac{V}{4\pi^2 m \hbar^3} \int_0^\infty dp\, \frac{p^5}{5} \left(-\frac{\partial}{\partial p} \underbrace{\langle n_{\vec{p}} \rangle}_{= \frac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1}} \right) =$$

$$= \frac{\beta V}{20\pi^2 m^2 \hbar^3} \int_0^\infty dp\, \frac{p^6 e^{\beta(\epsilon_{\vec{p}} - \mu)}}{\left(e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1\right)^2}.$$

$\frac{\partial}{\partial p} \langle n_{\vec{p}} \rangle$ is sharply peaked at $p = p_F$: expand as in computing the $f_{3/2}(z)$ expansion for $z \gg 1$.

Then $$\mathcal{U} \simeq \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]. \quad (\circledast)$$

Now, consider $$\sum_{|\vec{p}| < p_F} \frac{p^2}{2m} = \frac{V}{(2\pi\hbar)^3} \int_0^{p_F} dp\, p^4 \frac{4\pi}{2m} =$$

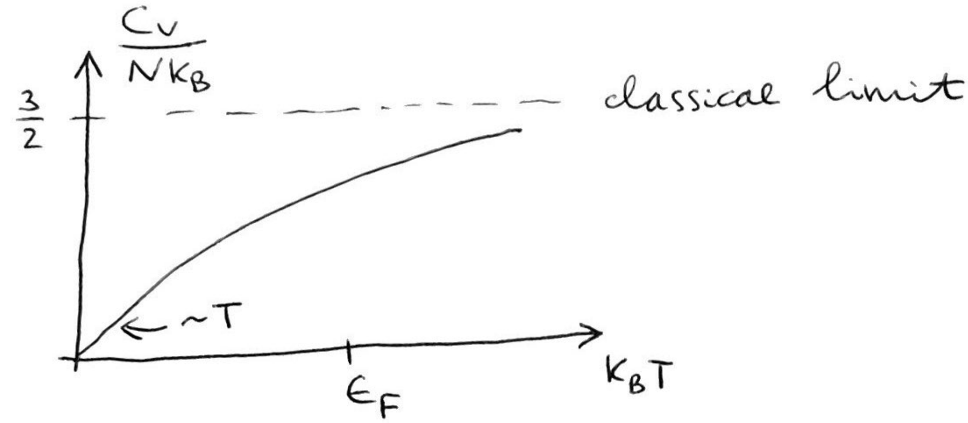
$$= \frac{V}{(2\pi)^2 \hbar^3 m} \frac{p_F^5}{5} = \frac{1}{5(2\pi)^2 \hbar^3 \cancel{m}} \left(\underbrace{\frac{p_F^2}{2m}}_{\epsilon_F} 2\cancel{m} \right) \underbrace{p_F^3 V}_{N \left(\frac{3}{4\pi}\right)(2\pi\hbar)^3} \ominus$$

$$\ominus\; N \epsilon_F \frac{3}{2} \frac{2}{5} = \frac{3}{5} N \epsilon_F.$$

So, the 1st term in Eq. $(\circledast)$ is the ground-state energy of the degenerate Fermi gas @ $T = 0$.

Finally,

$$\frac{C_V}{Nk_B} \simeq \frac{3}{5}\epsilon_F \times \frac{5\pi^2}{12} 2 \frac{k_B T}{\epsilon_F} = \frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}.$$



Only particles within $k_B T$ of ϵ_F can get excited at low T:

$$\frac{k_B T}{\epsilon_F} N \approx \#\text{ excited particles}$$

Then $$\Delta U \sim \frac{k_B T}{\epsilon_F} N k_B T = N\epsilon_F \left(\frac{k_B T}{\epsilon_F}\right)^2$$

Then $$\frac{C_V}{Nk_B} \sim \frac{k_B T}{\epsilon_F},$$ consistent with above

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The EoS is given by

$$p = \frac{2}{3}\frac{U}{V} \simeq \frac{2}{5}\frac{\epsilon_F}{v}\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right].$$

$p \neq 0$ even if $T=0$ (!), due to Pauli exclusion principle.