Lecture 11

Bose-Einstein condensation

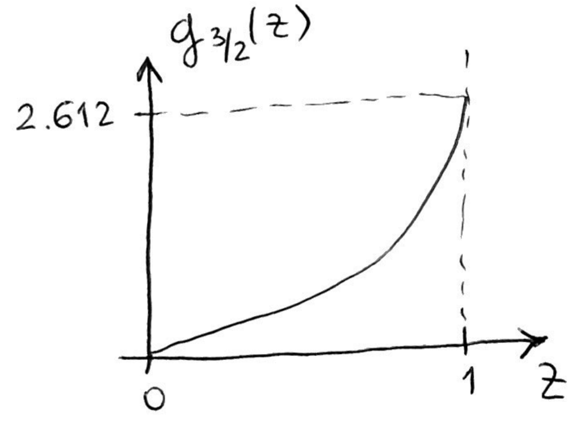
Recall that $\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V}\frac{z}{1-z}$, where

$$v = \frac{V}{N}, \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}} \Leftarrow \text{thermal wavelength}$$

Further, $$\underset{\substack{\uparrow \\ n=\frac{3}{2}\text{ here}}}{g_n}(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^n}.$$

For $g_{3/2}(z)$ to be bounded, we must have $z \le 1 \Rightarrow$ altogether, $\underbrace{0 \le z}_{\text{recall that } z = e^{\beta\mu}} \le 1$

$$g_{3/2}(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^{3/2}} = \underbrace{\zeta\left(\frac{3}{2}\right)}_{\text{Riemann zeta function}} \simeq 2.612$$



$0 \le z \le 1$:

$$g_{3/2}(z) \le g_{3/2}(1)$$

Recall that $\frac{z}{1-z} = \underbrace{\langle n_0 \rangle}_{\langle occ \rangle \text{ of the } \vec{p}=0 \text{ level}}$, then

$$\frac{\lambda^3}{v} - g_{3/2}(z) = \frac{\lambda^3}{V}\langle n_0 \rangle .$$

If $\frac{\lambda^3}{v} > g_{3/2}(1) \Rightarrow \frac{\langle n_0 \rangle}{V} > 0$ always holds at this T, v

Bose-Einstein (BE) condensation
[finite fraction of all particles in the $\vec{p}=0$ state]

The system in this region (called the condensation region) is a mixture of 2 phases, one with $\vec{p}=0$ and the other with $\vec{p} \neq 0$. The condensation region is bounded by $\underbrace{\frac{\lambda^3}{v} = g_{3/2}(1)}_{\text{2D surface}}$ in the p-v-T space

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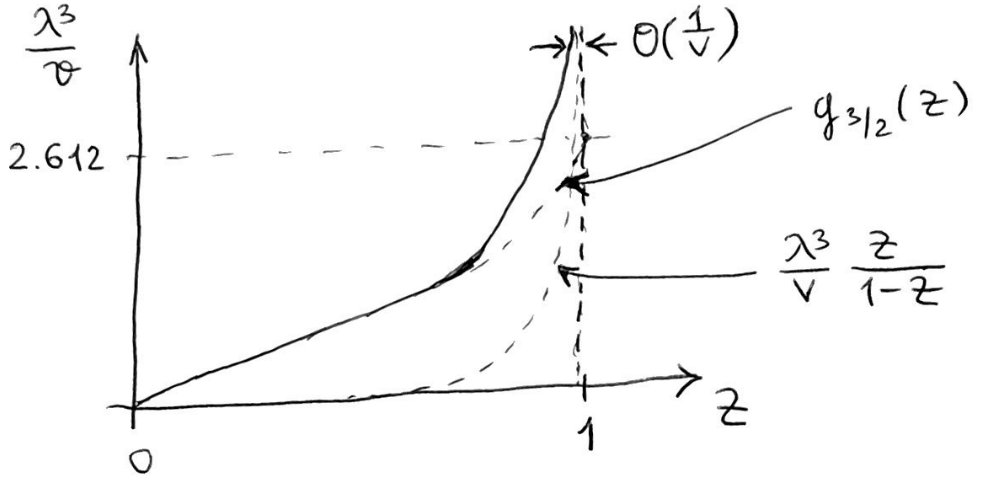
If v is given, $\lambda_c^3 = v\, g_{3/2}(1)$, or

$$k_B T_c = \frac{2\pi\hbar^2/m}{(v\, g_{3/2}(1))^{2/3}} .$$

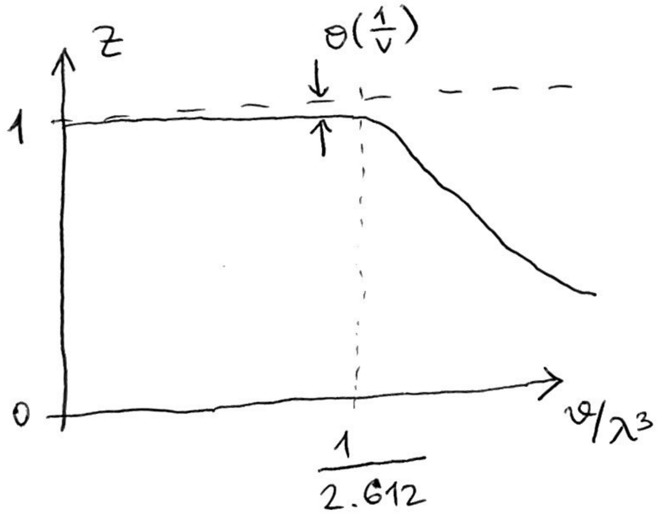
Note that $\lambda_c \sim v^{1/3}$, the average interparticle separation

If T is given, $v_c = \dfrac{\lambda^3}{g_{3/2}(1)}$.

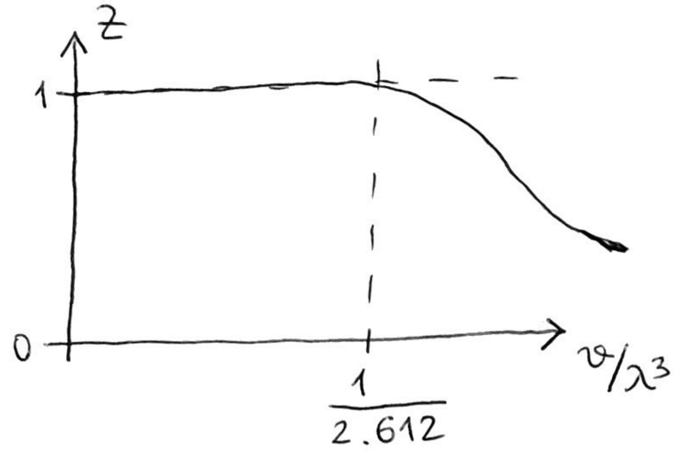
Condensation region: $\underbrace{T < T_c \text{ or } v < v_c}_{\frac{\lambda^3}{v} > g_{3/2}(1)}$ then



$$\frac{\lambda^3}{v} = g_{3/2}(z) + \frac{\lambda^3}{V}\frac{z}{1-z} \quad ; \quad V \text{ large but finite}$$



Finally, in the $V \to \infty$ limit we have:



$$z = \begin{cases} 1, & \frac{\lambda^3}{v} \geq g_{3/2}(1) \\ \text{root of } \frac{\lambda^3}{v} = g_{3/2}(z) & \text{otherwise} \end{cases}$$

Next, recall that

$$\frac{N}{V} = \frac{1}{V} \sum_{\vec{p} \neq 0} \langle n_{\vec{p}} \rangle + \frac{\langle n_0 \rangle}{V} .$$

For ex., $\frac{\langle n_1 \rangle}{V} = \frac{1}{V} \frac{1}{z^{-1} e^{\beta \epsilon_1 - 1}} \leq \frac{1}{V} \frac{1}{e^{\beta \epsilon_1 - 1}}$, where

$$2m\epsilon_1 = \left(\frac{2\pi\hbar}{L} \right)^2 \underbrace{(n_1^2 + n_2^2 + n_3^2)}_{l_1 = 1 \quad (\neq 0)} = (2\pi\hbar)^2 \frac{1}{V^{2/3}} .$$

$$V \to \infty : \quad \frac{\langle n_1 \rangle}{V} \leq \frac{1}{V} \frac{1}{\beta \epsilon_1} \sim \frac{1}{V^{1/3}} \to 0 .$$

Thus, occupancies are 'thinly spread' for $\forall \langle n_{\vec{p}} \rangle$ with $\vec{p} \neq 0$: only $\frac{\langle n_0 \rangle}{V}$ may be finite.

Now, consider

$$\frac{\langle n_0 \rangle}{N} = v\,\frac{\langle n_0 \rangle}{V} = 1 - \frac{v\, g_{3/2}(z)}{\lambda^3}.$$

Then $v = \text{const}$

$$\frac{\langle n_0 \rangle}{N} = 1 - \frac{\lambda_c^3}{g_{3/2}(1)}\,\frac{g_{3/2}(z)}{\lambda^3} = 1 - \frac{\lambda_c^3}{\lambda^3}\,\frac{g_{3/2}(z)}{g_{3/2}(1)}.$$

$\frac{\lambda^3}{v} \geq g_{3/2}(1)$ [condensation phase]

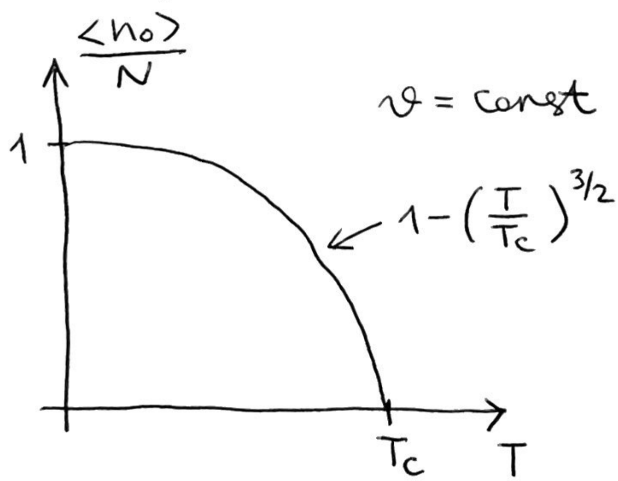
However, $z \to 1$ in this regime, s.t.

$$\frac{\langle n_0 \rangle}{N} = \underbrace{1 - \frac{\lambda_c^3}{\lambda}}_{} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \quad [v = \text{const}]$$

$T = \text{const}$:

$$\frac{\langle n_0 \rangle}{N} = 1 - \frac{v\, g_{3/2}(z)}{v_c\, g_{3/2}(1)} \underset{z \to 1}{=} 1 - \frac{v}{v_c} \quad [T = \text{const}]$$

If $\frac{\lambda^3}{v} < g_{3/2}(1)$, we are not in the condensation phase and $\frac{\langle n_0 \rangle}{N} = 0$.



$T = 0$: all particles in the $\vec{p} = 0$ state

Eq'n of state:

$$\frac{p}{k_BT} = \begin{cases} \frac{1}{\lambda^3} g_{5/2}(z) & (v > v_c, \text{ no condensation}) \\ \frac{1}{\lambda^3} g_{5/2}(1) & (v < v_c, \text{ condensation}) \end{cases}$$

(*)

$\boxed{g_{5/2}(1) = \zeta\left(\frac{5}{2}\right) \simeq 1.342}$

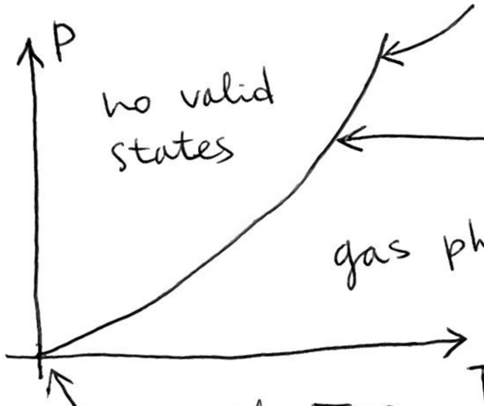
What about $\frac{1}{V}\log(1-z)$?

<u>No condensation</u>: $z<1 \Rightarrow \frac{1}{V}\log(1-z) \to 0$ as $V\to\infty$

<u>Condensation</u>: $1-z = \mathcal{O}\left(\frac{1}{V}\right)$ as discussed above,

s.t. $\frac{1}{V}\log(1-z) \sim \frac{1}{V}\log\frac{1}{V} \to 0$ as $V\to\infty$

Eq. (*) shows that p does not depend on v if $v<v_c$.



condensed phase ($\vec{p}=0$ fraction + gas fraction)

$p = \frac{k_BT}{\lambda^3} g_{5/2}(1) = g_{5/2}(1)\left(k_BT\right)^{5/2} \times \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \sim T^{5/2}$

$p=0$ if $T=0$ in the condensed phase (only condensed fraction remains) $\Rightarrow$ condensed phase exerts no pressure

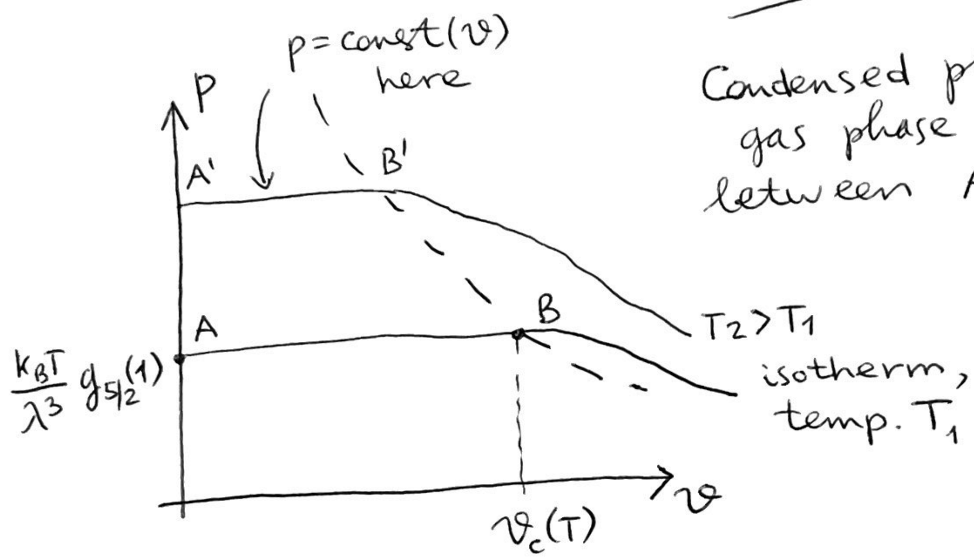
In the p-v coordinates:

$$\frac{p}{k_B T_c} = \frac{g_{5/2}(1)}{\lambda_c^3} \quad \text{at the transition line } (v = v_c)$$

But then

$$p = g_{5/2}(1)\,\frac{1}{v\,g_{3/2}(1)}\;\frac{2\pi\hbar^2/m}{[v\,g_{3/2}(1)]^{2/3}}\,, \text{ or}$$

$$p\,v^{5/3} = \frac{2\pi\hbar^2}{m}\,\frac{g_{5/2}(1)}{[g_{3/2}(1)]^{5/3}}\,.$$



Condensed phase and gas phase coexist between A & B, A' & B', etc.

Vapor (gas) pressure in the coexistence phase: $p(T) = \frac{k_BT}{\lambda^3}\,g_{5/2}(1)$.

$$\text{Then}\quad \frac{dp}{dT} = \frac{k_B}{\lambda^3}\,g_{5/2}(1) - \frac{3k_BT}{\lambda^4}\,\overbrace{\frac{d\lambda}{dT}}^{-\frac{1}{2T}\lambda}\,g_{5/2}(1) =$$

$$= \frac{5}{2}\,\frac{k_B}{\underbrace{\lambda^3}_{=v_c\,g_{3/2}(1)}}\,g_{5/2}(1) = \frac{1}{T v_c}\left[\frac{5}{2}\,k_BT\,\frac{g_{5/2}(1)}{g_{3/2}(1)}\right].$$

Recall the Clausius-Clapeyron equation:

$$\frac{dp(T)}{dT} = \frac{\Delta S}{\Delta v} = \frac{L}{T\Delta v} \quad \text{, where}$$

$L = T\Delta S$ is the latent heat of transition

Here, $\Delta v = v_c$ since $v = v_c$ in the gas phase, $v = 0$ in the condensed phase

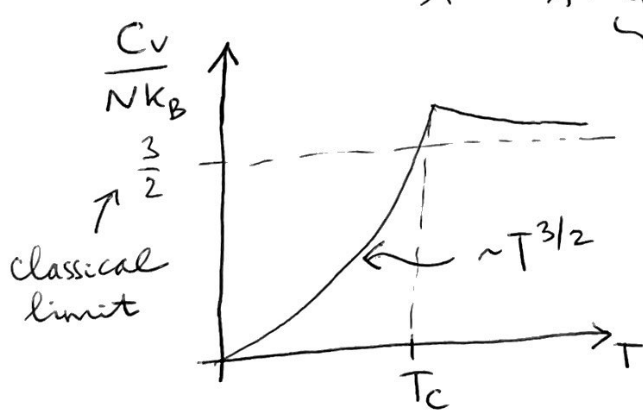
Then $L = \frac{5}{2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$ is the latent heat $\Rightarrow$ BE condensation is a 1st order phase transition.

Finally, recall that

$$\frac{U}{V} = \frac{3}{2}\frac{k_B T}{\lambda^3} g_{5/2}(z) \Rightarrow \frac{U}{N} = \begin{cases} \frac{3}{2}\frac{k_B T v}{\lambda^3} g_{5/2}(z), & T > T_c \\ \frac{3}{2}\frac{k_B T v}{\lambda^3} g_{5/2}(1), & T < T_c \end{cases}$$

Then

$$\frac{C_V}{N k_B} = \begin{cases} \frac{3}{2} v \frac{d}{dT}\left[\frac{T}{\lambda^3} g_{5/2}(z)\right], & T > T_c \\ \frac{3}{2} v \underbrace{\frac{d}{dT}\left[\frac{T}{\lambda^3}\right]}_{\frac{1}{\lambda^3} - \frac{3T}{\lambda^4}\underbrace{\frac{d\lambda}{dT}}_{-\frac{\lambda}{2T}} = \frac{5}{2}\frac{1}{\lambda^3}} g_{5/2}(1) = \frac{15}{4} v \frac{g_{5/2}(1)}{\lambda^3} \sim T^{3/2}, & T < T_c \end{cases}$$



BE system @ low T: liquid He^4.

$T_c = 2.18\,K \Rightarrow He^4$ exhibits the λ transition (BE condensation modified by intermolecular interactions)

No such transition occurs in He^3 which obeys Fermi statistics.

Ideal Bose gas: substitute m_{He^4} and ρ_{He^4} into

$$k_B T_c = \frac{2\pi\hbar^2/m}{\left[\rho^{-1} g_{3/2}(1)\right]^{2/3}}$$

to obtain $T_c = 3.14\,K$, off by ~50%.

For $T > T_c$, $g_{3/2}(z) = \frac{\lambda^3}{v}$.

Then

$$\frac{d}{dT}\left[\frac{T}{\lambda^3} g_{5/2}(z)\right] = \underbrace{\frac{d}{dT}\left[\frac{T}{\lambda^3}\right]}_{\frac{5}{2\lambda^3}} g_{5/2}(z) + \frac{T}{\lambda^3}\frac{d g_{5/2}(z)}{dT} = \frac{5}{2\lambda^3} g_{5/2}(z) + \frac{T}{\lambda^3} \underbrace{\frac{d g_{5/2}(z)}{dz}}_{} \frac{dz}{dT} \circleddash$$

use $g_{3/2}(z) = z \frac{d g_{5/2}}{dz}$

$$\frac{1}{v}\frac{d\lambda^3}{dT} = \frac{3\lambda^2}{v} \overbrace{\frac{d\lambda}{dT}}^{-\frac{\lambda}{2T}} = -\frac{3\lambda^3}{2T}\frac{1}{v} \circleddash -\frac{3 g_{3/2}}{2T}.$$

Now,

$$\frac{dz}{dT} = \underbrace{\frac{dz}{dg_{3/2}}}_{\left(\frac{g_{1/2}}{z}\right)^{-1}} \overbrace{\frac{dg_{3/2}}{dT}}^{} \approx -\frac{3z}{2T}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$g_{1/2}(z) = z\frac{dg_{3/2}}{dz} \rightarrow$

$$\Leftrightarrow \frac{5}{2\lambda^3} g_{5/2}(z) + \frac{\cancel{T}}{\lambda^3} \frac{g_{3/2}(z)}{\cancel{z}} \left(- \frac{3\cancel{z}}{2\cancel{T}} \frac{g_{3/2}(z)}{g_{1/2}(z)} \right) =$$

$$= \frac{5}{2\lambda^3} g_{5/2}(z) - \frac{3}{2\lambda^3} \frac{g_{3/2}^2}{g_{1/2}} .$$

Finally, at $T > T_c$

$$\frac{C_v}{N k_B} = \frac{3}{2} v \frac{d}{dT} [\ldots] = \frac{15}{4} \frac{v}{\lambda^3} g_{5/2}(z) -$$

$$- \frac{9}{4} \underbrace{\frac{v}{\lambda^3}}_{g_{3/2}^{-1}} \frac{g_{3/2}^2}{g_{1/2}} = \frac{15}{4} \frac{v}{\lambda^3} g_{5/2}(z) - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)},$$

as desired.