HW #5 solutions

(1.) For simplicity, assume $\vec{H}_0 = H_0 \hat{z}$, then

$$\int_\Omega d\Omega\, e^{\beta \vec{H}_0 \cdot \vec{S}} = 2\pi \int_0^\pi d\theta \sin\theta\, e^{\beta H_0 \cos\theta} =$$

$$= 2\pi \int_{-1}^{1} dx\, e^{\beta H_0 x} = \frac{4\pi}{\beta H_0} \sinh(\beta H_0).$$

Then $F_0 = -k_B T \log Z_0 = -k_B T N \log\left(\frac{4\pi}{\beta H_0} \sinh(\beta H_0) \right)$.

Next, $\underset{N \langle S_z \rangle_0}{\overset{\shortparallel}{\langle M_z \rangle_0}} = -\left(\frac{\partial F_0}{\partial H_0}\right)_T = k_B T N \frac{\beta \cancel{H_0}}{\cancel{4\pi}} \times \frac{1}{\sinh(\beta H_0)} \otimes$

$$\otimes \left[\frac{\cancel{4\pi}}{\cancel{\beta H_0}} \beta \cosh(\beta H_0) - \frac{4\pi}{\beta H_0^2} \sinh(\beta H_0) \right] =$$

$$= N \left[\coth(\beta H_0) - \frac{1}{\beta H_0} \right].$$

Thus, $\langle S_z \rangle_0 = \coth(\beta H_0) - \frac{1}{\beta H_0}$.

Now, consider

$$\langle \mathcal{H} - \mathcal{H}_0 \rangle_0 = -J \frac{NZ}{2} \langle S_z \rangle_0^2 + N H_0 \langle S_z \rangle_0 .$$

Using $\quad \Phi = F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0$, we obtain

$$\frac{1}{N}\frac{\partial \Phi}{\partial H_0} = \underbrace{\frac{1}{N}\frac{\partial F_0}{\partial H_0}}_{-\langle S_z \rangle_0} - J z \langle S_z \rangle_0 \frac{\partial \langle S_z \rangle_0}{\partial H_0} + \langle S_z \rangle_0 + H_0 \frac{\partial \langle S_z \rangle_0}{\partial H_0} = 0 \,, \text{ or}$$

$$H_0 = J z \langle S_z \rangle_0 \,.$$

In general, $\quad \vec{H}_0 = J z \langle \vec{S} \rangle_0$, as desired.

Finally, $\quad F_{mf} \underset{\uparrow \, \vec{H}_0 = H_0 \hat{z}}{=} -k_B T \log Z_0 - \frac{J z N}{2} \langle S_z \rangle_0^2 +$

$$+ N \langle S_z \rangle_0 \underbrace{J z \langle S_z \rangle_0}_{H_0} =$$

$$= -k_B T \log Z_0 + \frac{J z N}{2} \langle S_z \rangle_0^2 \,.$$

In general,

$$F_{mf} = -k_B T \log Z_0 + \frac{J z N}{2} \langle \vec{S} \rangle_0 \cdot \langle \vec{S} \rangle_0 \,, \text{ as desired}$$

$\boxed{\text{QED}}$

2. $\mathcal{H} = -J \sum_{i=1}^{N} S_i S_{i+1}$; $S_i = 0, \pm 1$

periodic BCs : $S_{N+1} = S_1$

Transfer matrix:

$$
\begin{array}{cccc}
 & S_{i+1}=1 & S_{i+1}=0 & S_{i+1}=-1 \\
T = & \left(\begin{array}{c} e^{\beta J} \\ 1 \\ e^{-\beta J} \end{array} \right. & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \left. \begin{array}{c} e^{-\beta J} \\ 1 \\ e^{\beta J} \end{array} \right)
\end{array}
\begin{array}{l} S_i=1 \\ S_i=0 \\ S_i=-1 \end{array}
$$

Acc. to Mathematica, the largest eigenvalue

$$\lambda_0 = \frac{e^{-\beta J}}{2}\left[1+e^{\beta J}+e^{2\beta J}+\sqrt{1-2e^{\beta J}-2e^{3\beta J}+11e^{2\beta J}+e^{4\beta J}}\right] =$$

$$= \frac{1}{2}\left[1+2\cosh(\beta J)+\sqrt{(e^{\beta J}+e^{-\beta J})^2-2e^{\beta J}-2e^{-\beta J}+9}\right] =$$

$$= \frac{1}{2}\left[1+2\cosh(\beta J)+\sqrt{4\cosh^2(\beta J)-4\cosh(\beta J)+9}\right] =$$

$$= \frac{1}{2}\left[1+2\cosh(\beta J)+\sqrt{(2\cosh(\beta J)-1)^2+8}\right].$$

Now, $f = -k_B T \log \lambda_0$ as $N \to \infty$.

$\beta \to 0$ $(T \to \infty)$: $\lambda_0 \to \frac{1}{2}[1+2+\sqrt{9}] = 3$, s.t.

$f \to -k_B T \log 3$ as expected.

$\beta \to \infty$ $(T \to 0)$: $\lambda_0 \to \frac{1}{2}[e^{\beta J}+\sqrt{e^{2\beta J}}] = e^{\beta J}$, s.t.

$f \to -k_B T \log(e^{\beta J}) = -J$, as expected.

$\boxed{\text{QED}}$

(3.) Recall that as $N \to \infty$,

$$f = -k_B T \log \left\{ e^{\beta J} \cosh(\beta H) + \right.$$

$$\left. + \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} \right\}, \text{ or}$$

$$\bar{f} = -\log\left\{ e^{K} \cosh(h) + \sqrt{e^{2K} \sinh^2(h) + e^{-2K}} \right\} =$$

$$= -K - \log\left\{ \cosh(h) + \sqrt{\sinh^2(h) + x} \right\}.$$

$$\begin{cases} x = e^{-4K} \to 0, \\ h \to 0 \end{cases} \quad \text{yield}$$

$$\bar{f} \to -K - \log\left\{1 + \sqrt{h^2 + x}\right\} \simeq -K - \sqrt{x}\sqrt{1 + \frac{h^2}{x}}.$$

The singular part of $\bar{f}$ is

$$\bar{f}_s = -\sqrt{x}\underbrace{\sqrt{1 + \frac{h^2}{x}}}_{-\bar{f}_s(1, \frac{h}{\sqrt{x}})} \sim b^{-\overset{d}{\underset{1}{\shortparallel}}} \bar{f}_s(b^{y_1} x, b^{y_2} h)$$

here

Thus, $b = \frac{1}{\sqrt{x}} \Rightarrow b^2 x = 1$.

$$\bar{f}_s \sim b^{-1} \bar{f}_s(\underbrace{b^2 x}_{=1}, \underbrace{b h}_{\frac{h}{\sqrt{x}}}) \Rightarrow \begin{cases} y_1 = 2, \\ y_2 = 1. \end{cases}$$

Next,

$$\langle s\rangle = \frac{e^{\beta J}\sinh(\beta H)}{\sqrt{e^{2\beta J}\sinh^2(\beta H) + e^{-2\beta J}}} =$$

$$= \frac{\sinh(h)}{\sqrt{\sinh^2(h) + x}} \to \frac{h}{\sqrt{h^2 + x}} =$$

$$= (\sqrt{x})^0 \frac{1}{\sqrt{1 + \frac{h^2}{x}}} = b^0 \langle s\rangle(\underbrace{b^2 x}_{=1}, \underbrace{bh}_{\frac{h}{\sqrt{x}}})$$

This is consistent with

$$\langle s\rangle \sim \left(\frac{\partial \bar{f}}{\partial h}\right)_x \sim b^{\underbrace{-d+y_2}_{=0 \text{ here}}} \langle s\rangle(b^{y_1} x, b^{y_2} h),$$

with $\begin{cases} y_1 = 2, \\ y_2 = 1 \end{cases}$ from above.

$\boxed{\text{QED}}$