HW#4 solutions

(1.) At equil., the pressure p will be the same in both compartments.

(a) $T=0$

$\mathcal{U} = \frac{3}{5} N \epsilon_F$, where

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{(2S+1)} n\right)^{2/3}$$

$$n = \frac{N}{V} = \frac{1}{v}$$

Then $p = -\left(\frac{\partial \mathcal{U}}{\partial V}\right)_S = \underset{\substack{\uparrow \\ \text{const}(V, S)}}{C} \frac{n^{5/3}}{(2S+1)^{2/3}}$ (S ← spin)

$p_1 = p_2$ gives $\left(\frac{n_1}{n_2}\right)^{5/3} = \frac{(2S_1+1)^{2/3}}{(2S_2+1)^{2/3}}$, or

$$\frac{V_2}{V_1} = \frac{(2S_1+1)^{2/5}}{(2S_2+1)^{2/5}} = \frac{2^{2/5}}{4^{2/5}} = \underline{\underline{2^{-2/5}}}.$$

(b) $T=\infty$

Both gases are classical ideal gases $\Rightarrow$ $p = nk_BT$ holds, and

$$\frac{n_1}{n_2} = \frac{V_1}{V_2} = \underline{\underline{1}}.$$

2. (a) $\log \Xi = -\sum_{\vec{p}} \log(1 - \underset{\uparrow \text{ fugacity}}{z}\, e^{-\beta \epsilon_{\vec{p}}}) =$

$$= -\frac{2\pi L^2}{h^2} \int_0^\infty dp\, p \log(1 - z e^{-\beta\epsilon_{\vec{p}}}), \text{ where}$$

$$\epsilon_{\vec{p}} = \frac{p^2}{2m} \quad \text{and} \quad L^2 = \underbrace{A}_{\text{area}}$$

(b) Recall that

$$\frac{P}{k_B T} = \frac{\log \Xi}{\underbrace{L^2}_{\substack{V \text{ in 3D,} \\ A \text{ in 2D}}}} = -\frac{1}{2\pi\hbar^2} \int_0^\infty dp\, p\, \underbrace{\log(1 - z e^{-\beta p^2/2m})}_{\substack{\text{expand} \\ \text{in powers} \\ \text{of } z}} \quad \text{!!}$$

$$= \frac{1}{2\pi\hbar^2} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^\infty dp\, p \left(z e^{-\beta p^2/2m}\right)^k \boxed{=}$$

$\uparrow$

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\boxed{=} \frac{1}{2\pi\hbar^2} \sum_{k=1}^{\infty} \frac{z^k}{k} \underbrace{\int_0^\infty dp\, p\, e^{-k\beta p^2/2m}}_{\frac{1}{2}\frac{2m}{k\beta} = \frac{m}{k\beta}} =$$

$$= \underbrace{\frac{m k_B T}{2\pi\hbar^2}}_{\lambda^{-2}} \underbrace{\sum_{k=1}^{\infty} \frac{z^k}{k^2}}_{g_2(z)} = \frac{g_2(z)}{\lambda^2}.$$

(c) Next,

$$n = \frac{N}{L^2} = z\frac{\partial}{\partial z}\left(\frac{\log \Sigma}{L^2}\right) = z\frac{\partial}{\partial z}\left(\frac{g_2(z)}{\lambda^2}\right) =$$

$$= \frac{1}{\lambda^2}\, z \sum_{k=1}^{\infty} \frac{z^{k-1}}{k} = \frac{1}{\lambda^2} \underbrace{\sum_{k=1}^{\infty} \frac{z^k}{k}}_{-\log(1-z)} = -\frac{\log(1-z)}{\lambda^2}.$$

Finally, $\log(1-z) = -\lambda^2 n,$

$$\underbrace{z = 1 - e^{-\lambda^2 n}}_{z(n,T)}.$$

$\uparrow$ through λ