Toric Code

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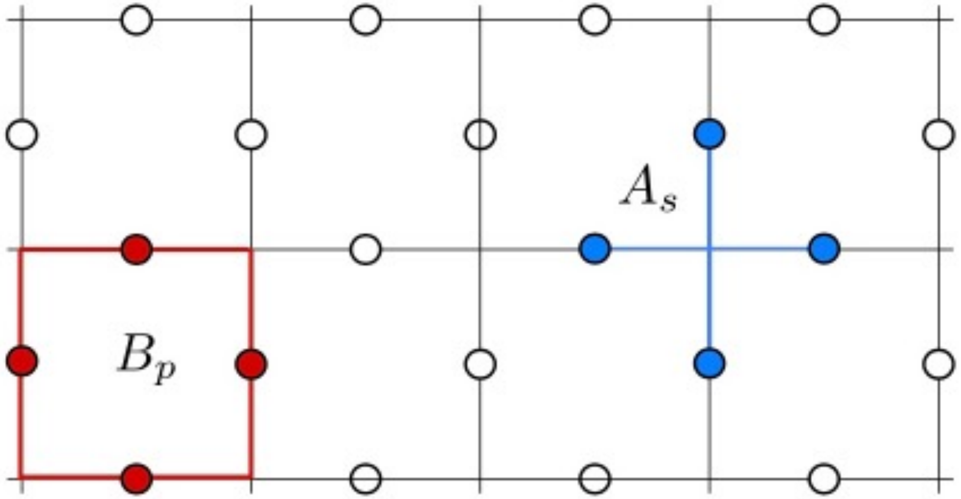
Outline

- Toric Code
  - Operator and System
  - Ground States
  - Excited States
  - Application to Quantum Information

- Experimental Realizations
  - Ground State
  - Entropy Measurement
  - Excitations
  - Error Correction

Operators and System



- $k \times k$ square lattice on a torus
- Each edge has a site with spin (use $|\uparrow\rangle = |\downarrow\rangle$)
- For each vertex s and face p we associate operators

$$A_s = \prod_{j \in star(s)} \sigma_j^x \qquad B_p = \prod_{j \in boundary(p)} \sigma_j^z$$

- These operators commute as $\{\sigma_l^i, \sigma_{l'}^j\} = 2\delta_{l,l'}\delta^{i,j}$ and $(\sigma_l^i)^2 = 1$
  - At most, two edges overlap
- A_s flips spins as $\sigma_j^x|\uparrow\rangle = |\downarrow\rangle$.
- The eigenvalues of these operators are +1 and -1

https://topocondmat.org/w12_manybody/topoorder.html

Toric Code Hamiltonian

On the lattice, we will consider the Hamiltonian

$$H_o = -\sum_s A_s - \sum_p B_p$$

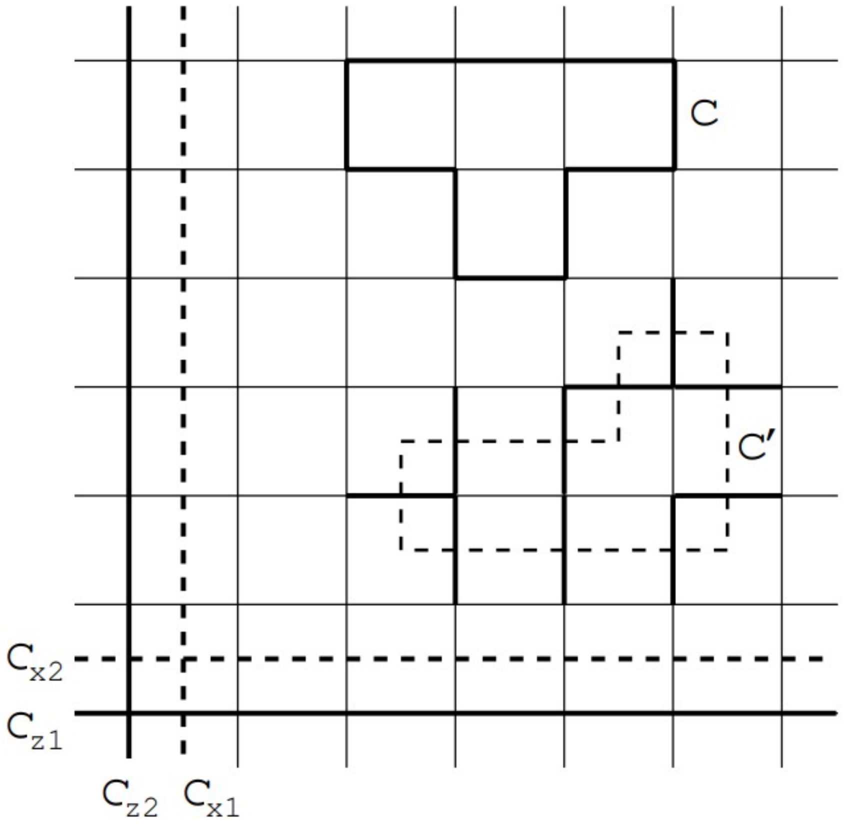
For the ground state $|G\rangle$,

$$A_s\,|G\rangle = |G\rangle \text{ and } B_p|G\rangle = |G\rangle \text{ for all p and s}$$

Ground States

- With only B_p, the ground state corresponds to loops c on the lattice.

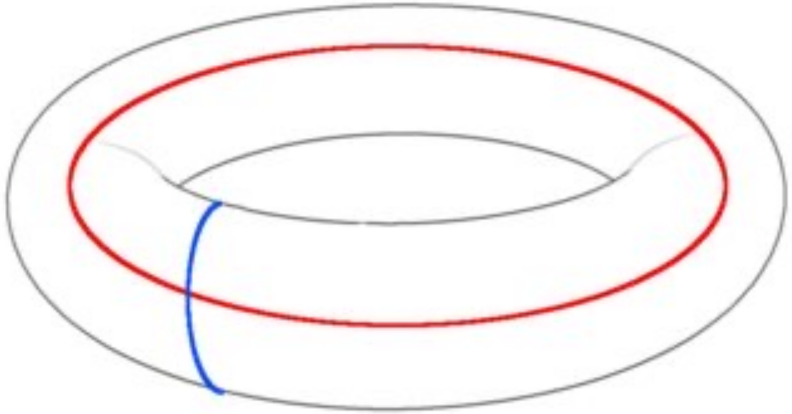
- With only A_s, the ground state corresponds to loops c' on the dual lattice.

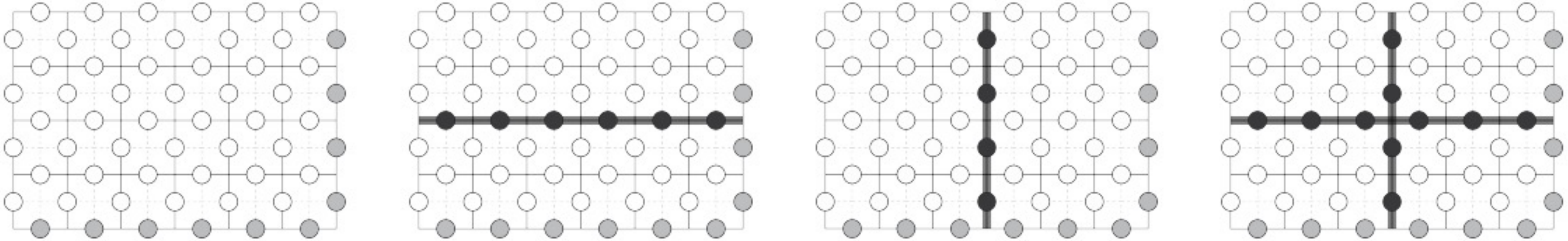


https://arxiv.org/pdf/quant-ph/9707021.pdf
https://phas.ubc.ca/~berciu/TEACHING/PHYS502/PROJECTS/20-Toric-PH.pdf

Ground States

- With both A_s and B_p we have 4-fold ground state of inequivalent loops
- These loops cannot be deformed into one another without breaking it





(a) (b) (c) (d)

https://phas.ubc.ca/~berciu/TEACHING/PHYS502/PROJECTS/20-Toric-PH.pdf

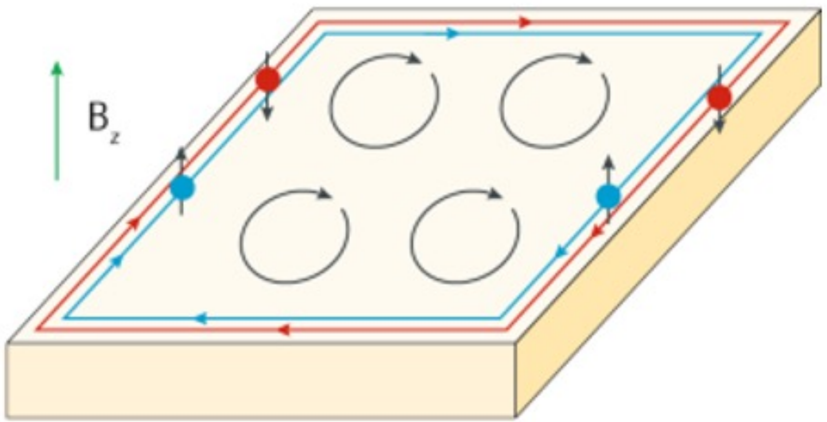
https://www.nature.com/articles/s42254-020-0209-1
https://www.sp.phy.cam.ac.uk/research/fundamentals-of-low-dimensional-semiconductor-systems/lowD

Topological Nuance

- The ground states are topologically protected
  - Cannot create another loop without destroying the loop
- The ground state is referred to as **topological order**

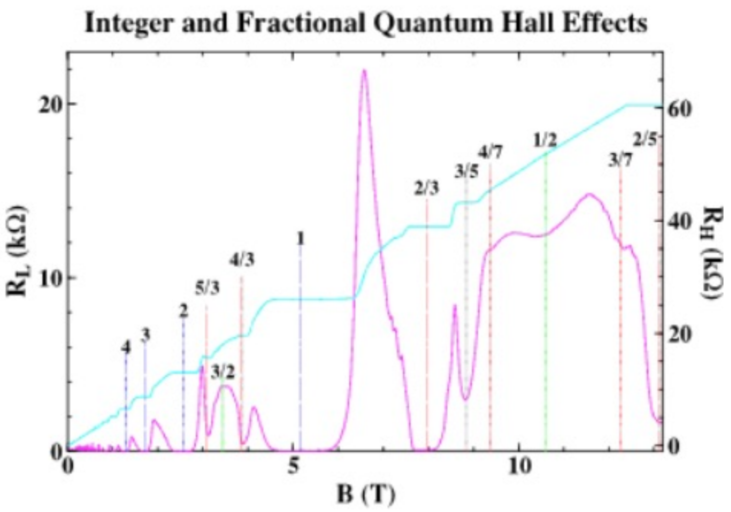
This is **not** related to the quantum Hall Effect/Chern number

- Protected surface state

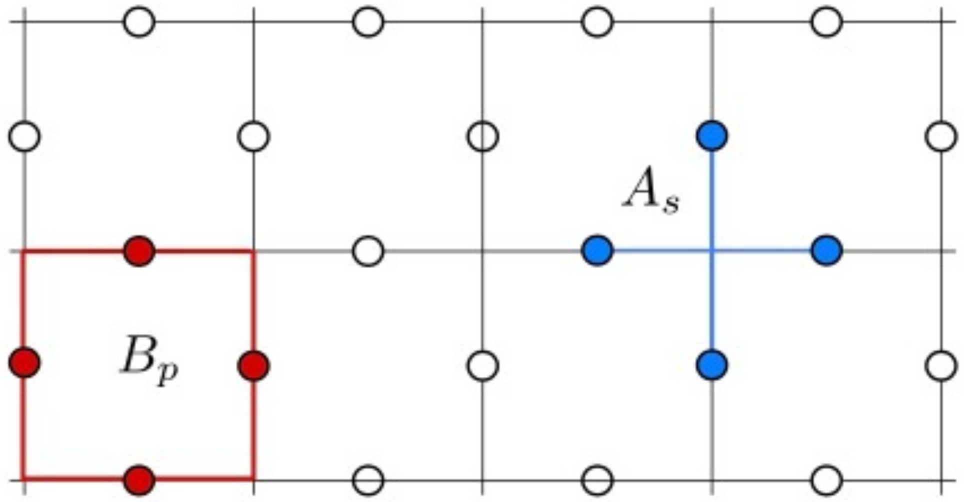


This **is** related to the fractional quantum Hall Effect

- Fractionalized/topological excitations



Operator Analogues



- Operators can be thought of as "monopole locators"
  - The operator B_p is a magnetic flux operator to measure magnetic monopoles (mod 2).
  - The operator A_s is an electric charge operator to measure electric monopoles (mod 2).

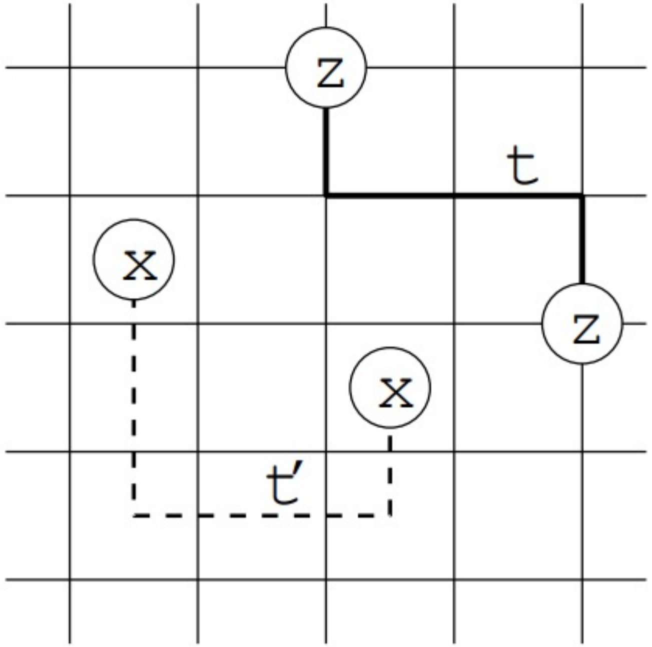
- A_s are $\mathbb{Z}_2$ gauge fields as $A_s^\dagger \sigma_l^z A_s = \rho_s \sigma_l^z$, $\rho_s = 1$ if A_s touches the site l and -1 otherwise.

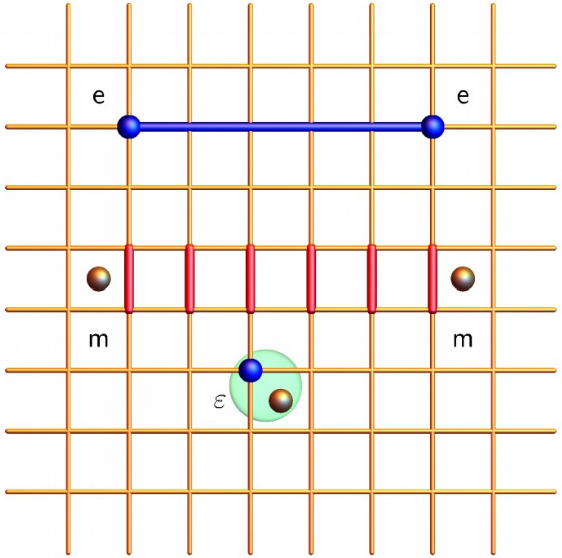
Excited Particle Pairs

We can't form one particle states, but we can produce two particle states with "string" operators

$$S^z(t) = \prod_{j\in t} \sigma_j^z \quad S^x(t') = \prod_{j\in t'} \sigma_j^x$$

- Particles are at the endpoints of the strings.
  - z-type "electric charges" (A_s "measures" – charge)
  - x-type "magnetic charges" (B_p "measures" – charge)
- The operators $S^z(t)$ and $S^x(t')$ commute with A_s and B_p except at endpoints.





https://arxiv.org/pdf/quant-ph/9707021.pdf
https://arxiv.org/pdf/1601.03742.pdf

Abharanov-Bohm Phase

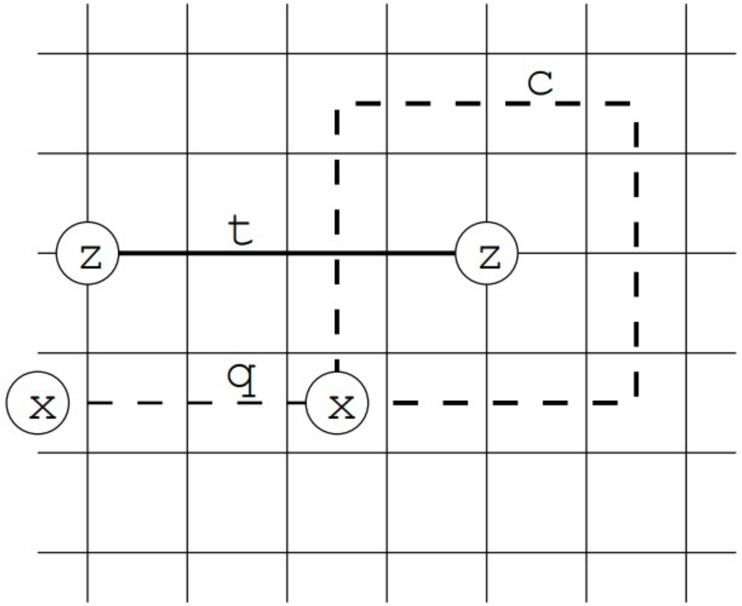
- What happens when we move different particles around one another?
- Consider a state with x-type charges $|\psi^x\rangle$
- Add z-type charges $\mathrm{S^z(t)}$ such that

$$|\psi_i\rangle = \mathrm{S^z(t)}|\psi^x\rangle$$

- Move a x-type charge around loop c with string operator $\mathrm{S^x(c)}$. Then

$$|\psi_f\rangle = S^x(c)\mathrm{S^z}(t)|\psi^x\rangle = -|\psi_i\rangle$$

Using that $S^x(c)$ and $\mathrm{S^z}(t)$ anticommute.



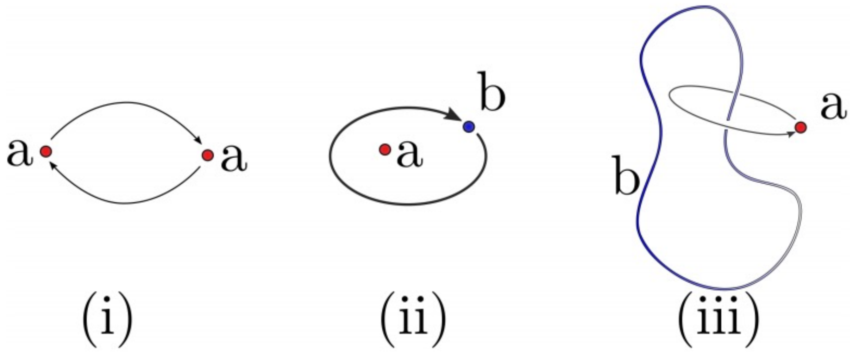
https://arxiv.org/pdf/quant-ph/9707021.pdf

Abelian Anyons

Typically for bosons and fermions, we have exchange rules

$$|x_1, x_2\rangle = e^{i\theta}|x_2, x_1\rangle$$

Where $\theta = \pi$ for fermions and $\theta = 2\pi$ for bosons
For a full rotation we return to $|x_1, x_2\rangle$



Here, we return to the initial state with phase factor -1.

Corresponds to anyons (this case visons) with arbitrary θ.

Manifestation of **topological order**:

- A hidden long-range order that can not be described by any local order parameter.

https://arxiv.org/pdf/1601.03742.pdf

Use in Quantum Information

- The ground state is protected from local perturbations
  - Ground states get projected to another ground state
- Ground states identified by loops
  - Treat as qubits (quantum bits)
    - $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - Have four-dimensional ground state or two qubits
  - String operators can be logic operations
- Anyon annihilation takes us between ground states

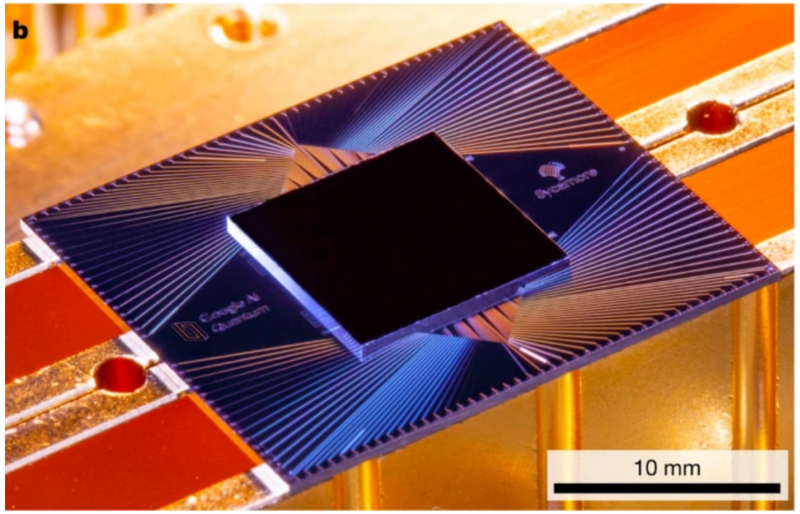
Toric Code Summary

- Local operators form long range order
- Ground State protected by topology (unique loops)
- Topological order implies excitations are anyons
- Form qubits from topologically identified ground state

https://www.nature.com/articles/s41586-019-1666-5

Topological Order on Quantum Processor

- Toric Code ground state constructed on Google's Sycamore quantum processor
  - Create lattice with 31 qubits
  - $A_s = \prod_{j \in star(s)} \sigma_j^z,\ B_p = \prod_{j \in boundary(p)} \sigma_j^x$
  - Ground state restricted to a single state



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Realizing topologically ordered states on a quantum processor

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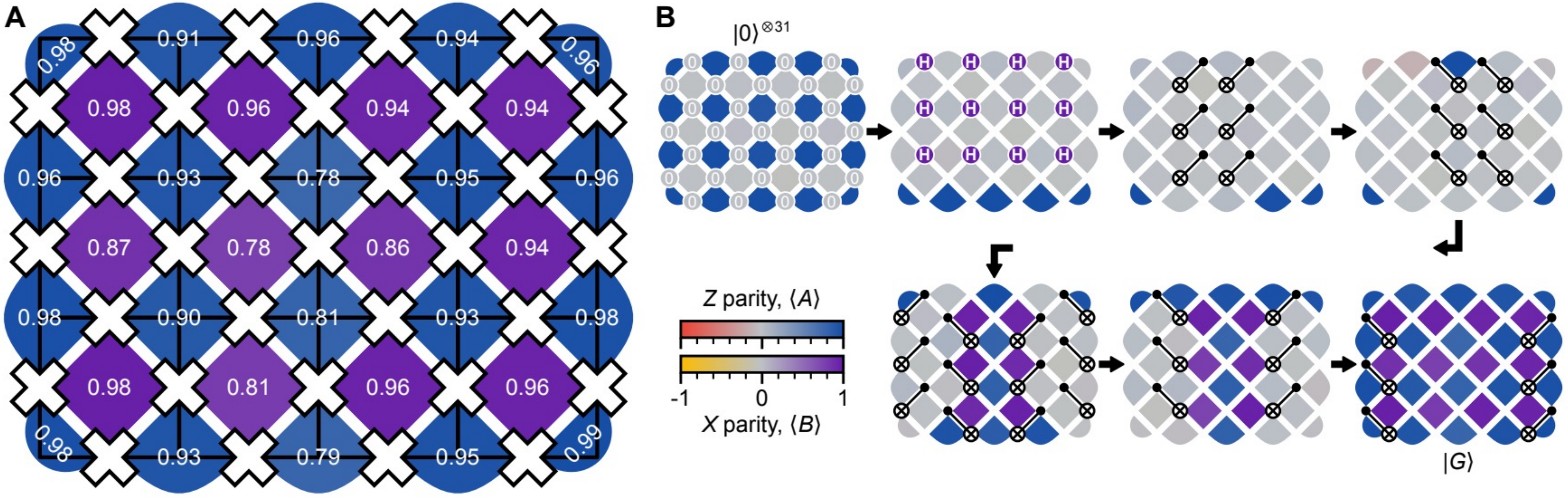
The discovery of topological order has revolutionized the understanding of quantum matter in modern physics and provided the theoretical foundation for many quantum error correcting codes. Realizing topologically ordered states has proven to be extremely challenging in both condensed matter and synthetic quantum systems. Here, we prepare the ground state of the toric code Hamiltonian using an efficient quantum circuit on a superconducting quantum processor. We measure a topological entanglement entropy near the expected value of ln 2, and simulate anyon interferometry to extract the braiding statistics of the emergent excitations. Furthermore, we investigate key aspects of the surface code, including logical state injection and the decay of the non-local order parameter. Our results demonstrate the potential for quantum processors to provide key insights into topological quantum matter and quantum error correction.

Different phases of matter can commonly be distinguished in terms of spontaneous symmetry breaking and local order parameters. However, several exotic quantum phases have been discovered in recent decades that defy this simple classification, instead exhibiting *topological order* [1, 2]. These phases are characterized by their long-range quantum entanglement and the emergence of quasiparticles with *anyonic* exchange statistics. Moreover, they have energetically gapped ground states with degeneracies that depend on their boundary conditions. The non-local nature of these states makes them particularly attractive platforms for fault tolerant quantum computation, as quantum information encoded in locally indistinguishable ground states is robust to local perturbations [3, 4]. This is the underlying principle of topological quantum error correcting codes, where the logical codespace corresponds to the degenerate ground state subspace of a lattice model [5–7].

An archetypical topological two-dimensional lattice model is the toric code, which exhibits so-called $\mathbb{Z}_2$ topological order [3]. The realization of the toric code on a plane—the surface code—has emerged as one of the most promising stabilizer codes for quantum error correction due to its amenable physical requirements [8]. Given both its inherent richness and quantum computing applications, experimentally realizing $\mathbb{Z}_2$ topological order has sparked extensive interest, resulting in several experimental studies with comparatively small-scale synthetic quantum systems [9–19]. Despite these efforts, the experimental realization of topologically ordered states remains a major challenge, requiring the generation of long-range entanglement. This can be achieved by identifying suitable quantum systems with topologically ordered ground states or by constructing a topologically ordered state in an engineered quantum system. Probing the non-local topological properties of such a state on an array of qubits requires high fidelity gates and a

Ground State Setup

- Start with $A_s = 1$ and $B_p = 0$ so that $|0\rangle^{\otimes 31}$
- Transform upper qubit to $1/\sqrt{2}(|0\rangle + |1\rangle)$
- Perform CNOT logic gate to require $A_s = 1$ and $B_p = 1$



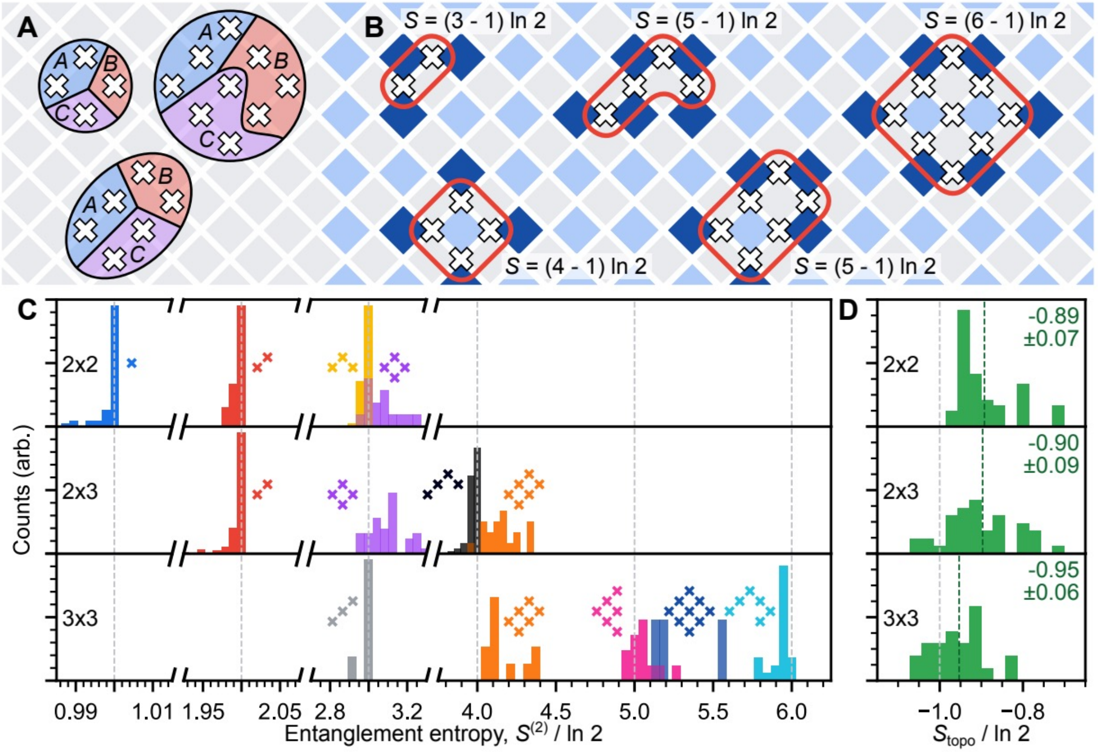
Entanglement Picture

- Represented in local basis ($|\uparrow\rangle, |\downarrow\rangle$)
  - Large entanglement of states

$$|0\rangle = \prod_p (1 + B_p)\, |0\rangle^{\otimes 31}$$

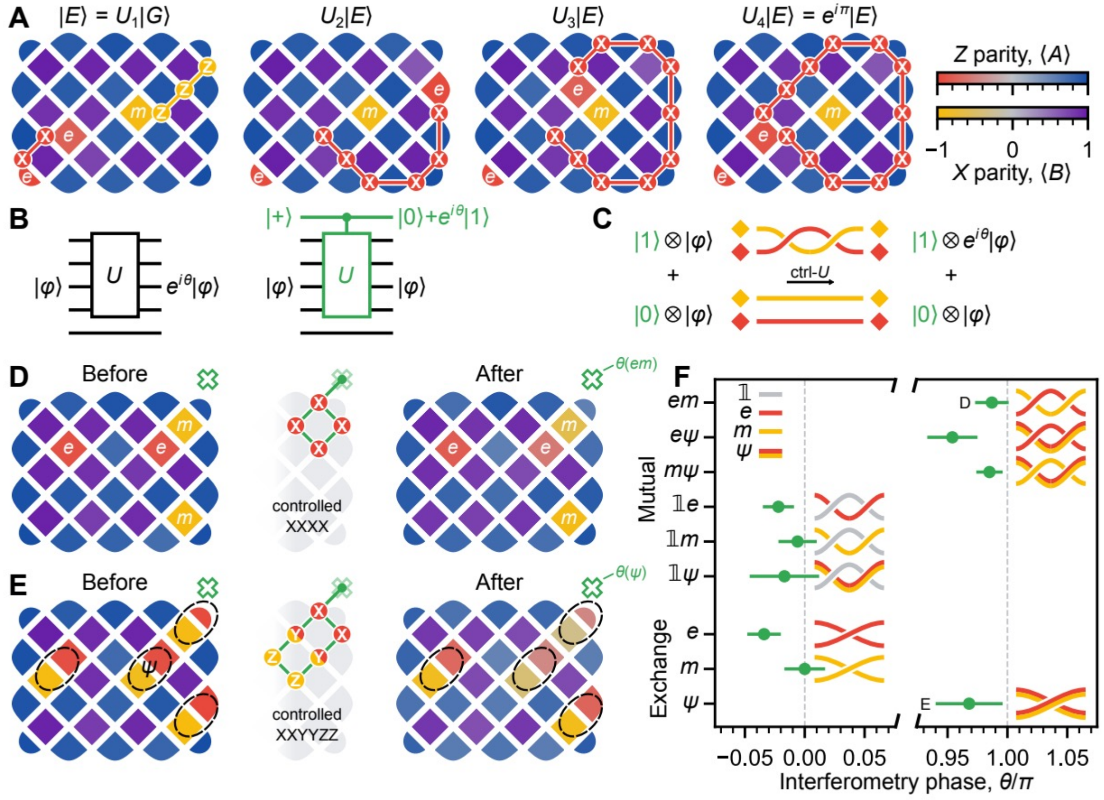
- Measure entanglement through entropy, S_{topo}
- Break system into areas and measure entropy
- For toric code

$$S_{topo} = -ln2$$



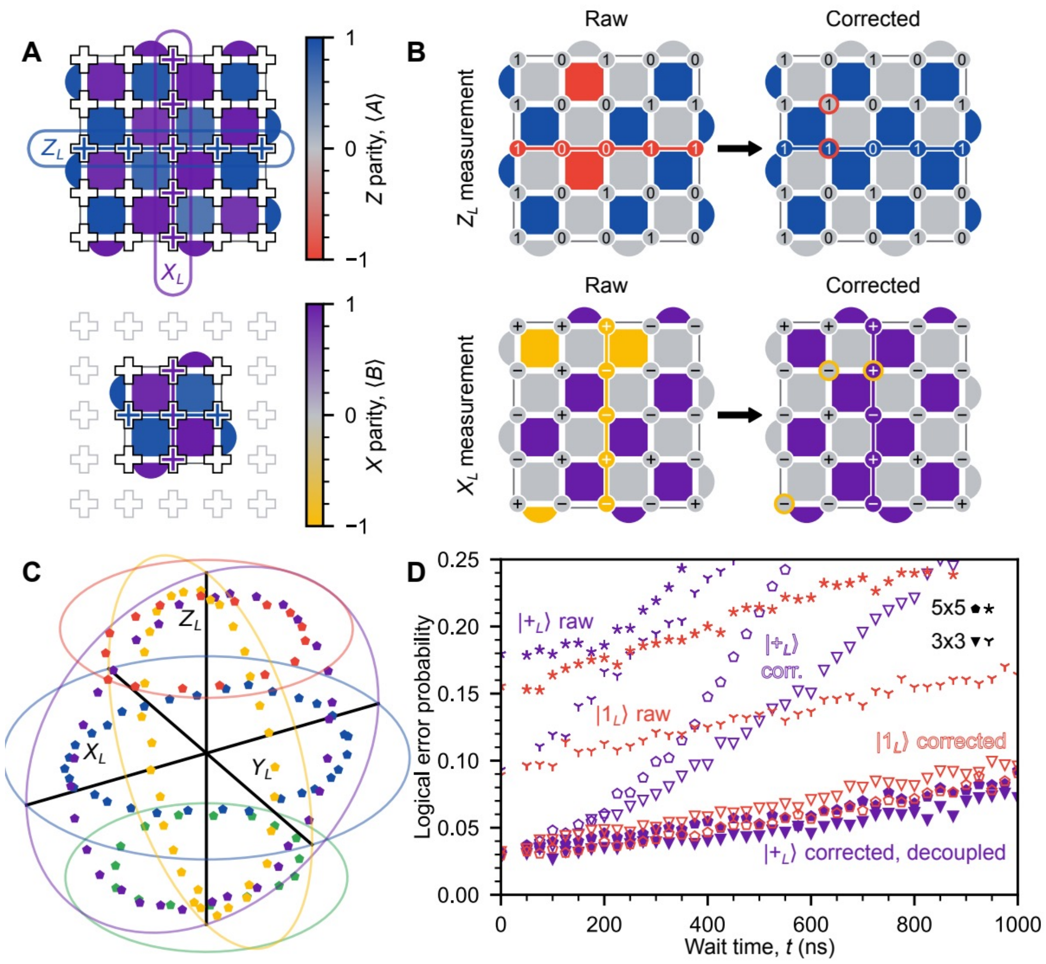
Excited State

- Applied String operators
- Moved anyons
- Ramsey interferometry to measure phase difference
- Tomography on auxiliary qubits
  - 18000 measurements



Error Correction

- Apply string operator along line
- Locate errors with A_s and B_p
- Identify lowest flips to correct
- Remeasure and compare error probability



Conclusion

- Toric code stabilizes states through local operators
- Ground state can be used as qubits
- Form ground state on quantum computer
- Excitations follow mutual statistics