The Toric Code

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We review the toric code, a simple spin-lattice model that exhibits topological order and anyonic excitations. We derive these phenomena from a model Hamiltonian, explain their significance, and highlight the connection to fault-tolerant quantum computation.

I. INTRODUCTION

Topology is the branch of mathematics that studies the properties of objects which do not change under smooth deformations, one classic example being the number of holes in a torus. In other words, topologists study global properties that are invariant to many local perturbations. From this perspective it is not hard to see why physicists are also interested in topology. In this article we review one example of a physical model where the ground state degeneracy and elementary excitations are determined by topology: the *toric code*. We will take a condensed-matter physicist's approach to this model, starting from the Hamiltonian and deriving the ground state. We also discuss the low-energy excitations and their unusual exchange statistics, and explain why the toric code is a cornerstone model in the study of fault-tolerant quantum computation.

II. SPIN HAMILTONIAN

Consider a 2D square lattice with periodic boundary conditions (i.e. a torus) and spin-$\frac{1}{2}$ particles located on each edge (Fig. 1). Define the following Hamiltonian:

$$H = -\sum_v A_v - \sum_p B_p, \tag{1}$$

where the first sum is over all vertices v and the second is over all faces or plaquettes p. The vertex operators A_v and plaquette operators B_p are tensor products of Pauli operators $X = \sigma^x$ and $Z = \sigma^z$ acting on individual

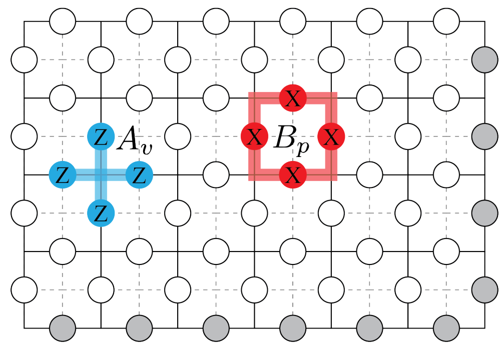

FIG. 1. Our model is a square lattice with periodic boundary conditions and a spin-$\frac{1}{2}$ particle (circles) on each edge. The dashed lines show the dual lattice. A_v and B_p consist of Pauli operators Z (blue) and X (red) centered around a vertex or plaquette of the primal lattice.

spins:

$$A_v = \prod_{j\in \mathrm{star}(v)} Z_j; \quad B_p = \prod_{j\in \mathrm{bdy}(v)} X_j \tag{2}$$

where star(v) is the set of edges adjacent to vertex v, and bdy(p) is the set of edges forming the boundary of the plaquette p (Fig. 1).

We can start by noting that A_v and B_p are Hermitian and square to the identity, therefore have eigenvalues ± 1. Furthermore, each of them commutes with all the others. To see this, note that the overlap of A_v with any B_p occurs at an even number of edges. The operators X and Z anticommute on any given edge, but on an even number of edges this effect cancels itself out. Therefore, the ground state(s) will be the simultaneous $+1$ eigenstate of all the A_v and B_p operators.

The next step is to figure out what kind of states satisfy the above constraints. Before we begin, let's compute how many states we expect to find so that we can be confident of the final answer. Each unit cell has two spins, and each spin occupies a Hilbert space of dimension 2, so one unit cell has a 4-dimensional Hilbert space. If there are N unit cells, the Hilbert space of the lattice has dimension 2^{2N}. Now, think of A_v and B_p as constraints on this Hilbert space; violating any constraint costs energy, so the ground state should obey every constraint. Finally, note that while there are N each of A_v and B_p, each type gives only $N-1$ independent constraints because the product of all A_v is the identity, and similarly for B_p. Putting everything together, we have $2N-2$ constraints, so we expect a GS manifold of dimension $2^{2N-(2N-2)} = 4$.

It will be useful to introduce some notation for constructing the ground state. We adopt the convention that $|0\rangle$ is the $+1$ eigenstate of Z and $|1\rangle$ is the -1 eigenstate. In this basis, X acts as a spin-flip operator, i.e. $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$. Taking the $|0\rangle, |1\rangle$ notation literally, we will say that an edge is "occupied" if the spin there is in the state $|1\rangle$, and "empty" if it is $|0\rangle$. Recall that we are looking for $+1$ eigenstates of all A_v and B_p, but for now let's focus only on A_v. Clearly the state with all edges unoccupied has eigenvalue $+1$ for all A_v. If a single spin is flipped, the resulting state will have eigenvalue -1 for the A_v immediately adjacent to that edge (Fig. 2a). In fact, anytime a vertex v is surrounded by an odd number of occupied edges, the corresponding A_v will have eigenvalue -1. The only states which do *not* have at least one such vertex are those where the occupied edges form closed loops, so the set of all such states forms the $+1$ eigenspace of the A_v operators (Fig. 2b).

Clearly there are many, many possible states with closed loops, so the ground state is massively degenerate if we only satisfy the A_v constraints. To fix this degeneracy we add the B_p terms to the picture. Each B_p is made up of X operators, so it will flip all the spins

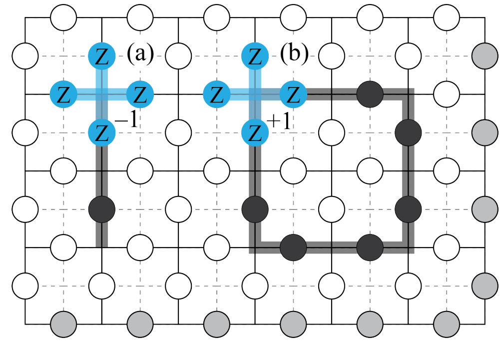

FIG. 2. Occupied edges are shaded black. (a) Applying A_v to the end of a string gives eigenvalue -1. (b) Applying A_v anywhere on a loop gives eigenvalue $+1$.

on the border of a given plaquette. This has the effect of adding a new loop of occupied edges or smoothly deforming an existing loop, in the sense that we cannot break or join a string of occupied edges by applying any combination of B_p. Notice that we can contract any loop that does not wind around the torus until it disappears completely; in this sense, such loops are equivalent to the trivial state with no loops. However, a little thought should convince you that a string of occupied edges that winds all the way around the torus in one direction cannot be contracted into nothing, as this would require breaking the string somewhere. This realization gives us 4 classes of loops which cannot be deformed into one another: the trivial class with no loops, a loop that winds around the torus in either direction, and the combination of two loops with one winding around in each direction (Fig. 3).

The fact that there are 4 inequivalent loop configurations should not be too surprising; as you may have guessed, they correspond to the 4-dimensional GS manifold. To see this, note that the 4 basic loops are not eigenstates of B_p by themselves, but a completely symmetric superposition of any one of them with all members of the trivial class *is* an eigenstate. In other words, we have a sort of "loop vacuum" (the trivial class), to which we can add zero, one, or two non-trivial loops. Each

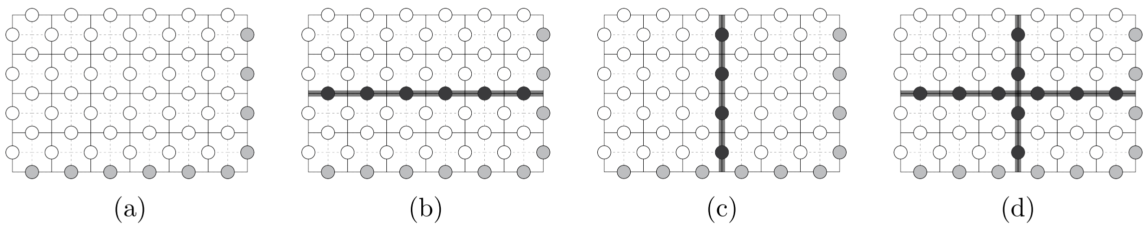

FIG. 3. Examples of the four classes of loops which cannot be deformed into each other.

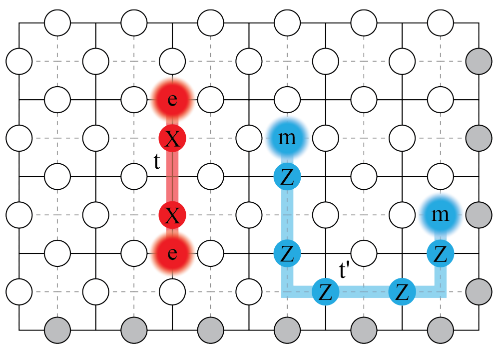

FIG. 4. Strings of X or Z operators applied to the ground state create pairs of “electric” or “magnetic” excitations e and m. The energy of each pair is $+2$ because the end of any string anticommutes with one A_v or B_p.

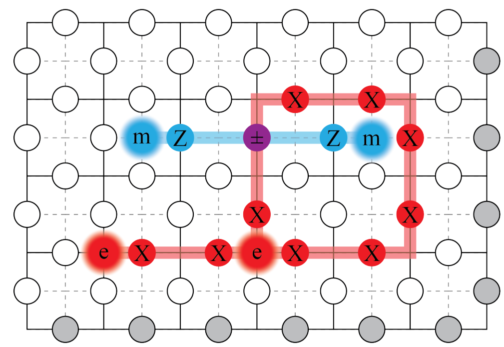

FIG. 5. Moving an e particle around an m particle causes the overall state to pick up a phase of -1, because the X and Z strings anticommute at exactly one site.

case is a degenerate eigenstate of the Hamiltonian (1), and together they form a basis for the 4-dimensional GS manifold. Note that this 4-fold degeneracy is directly related to the topology of the system (i.e. the periodic boundary conditions); we say that the toric code displays *topological order*.

III. ANYONIC EXCITATIONS

Now that we know the ground state, we can try to characterize the low-energy excitations. The easiest way to do this is to think in terms of the operators that create an excitation. These turn out to be *string operators* [1]:

$$S^x(t) = \prod_{j\in t} X_j; \quad S^z(t') = \prod_{j\in t'} Z_j, \tag{3}$$

where t (t') is a string of edges on the (dual) lattice (Fig. 4). If a string t is open-ended, the X operators at its endpoints will anticommute with one A_v each, raising the energy above the ground state by 2. Similarly, the ends of a dual string t' anticommute with B_p. Note that it is impossible to create an excitation with unit energy. To see this, suppose there is an operator C acting on a state $|\psi\rangle$ in the GS manifold such that C anticommutes with only one A_v (or B_p, it doesn't matter which). Recall that $\prod_v A_v = 1$, therefore

$$C|\psi\rangle = \prod_v A_v C|\psi\rangle = -C\prod_v A_v|\psi\rangle = -C|\psi\rangle,$$

i.e. $C|\psi\rangle = 0$. More intuitively, it is impossible to create an open string with one end, so the only allowed excitations come in pairs. We call the quasiparticle pairs created by $S^x(t)$ “electric charges” e and the pairs created by $S^z(t')$ “magnetic vortices” to match existing literature on gauge field models, but this is just a convention [1].

Perhaps the most interesting thing about

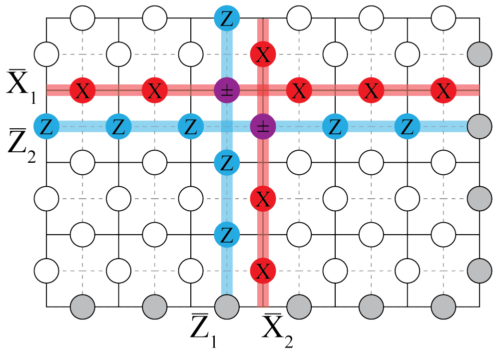

FIG. 6. Logical operators on acting on the qubits encoded by the ground state of the toric code. Anticommutation relations can be inferred from the intersection of the strings, and they match the usual Pauli operators on two qubits.

these quasiparticles is their exchange statistics. Clearly $S^x(t_1)$ commutes with $S^x(t_2)$ for any strings t_1 and t_2, as they only involve X terms. The same is true of S^z, so it appears that both the e and m particles are hard-core bosons. However, if we move an e particle around an m particle or vice versa (Fig. 5), it acquires a phase of -1 from the overlapping string operators. Quasiparticles that behave in this way are called anyons, because they fall outside the usual bosonic/fermionic picture [1]. Anyons are unique to 2D systems and feature in the theory of the fractional quantum Hall effect [2]. More recently, Bartolemi et al published strong experimental evidence for their existence using an "anyon collider" [3].

IV. FAULT-TOLERANT QUANTUM COMPUTATION

The ground state of the Hamiltonian (1) is protected from local perturbations. For instance, consider a local X perturbation $V = \lambda \sum_j X_j$ where j runs over every spin on the lattice and $\lambda \ll 1$. This perturbation will split the degeneracy of the ground state, but the splitting terms obtained from perturbation theory are of the form $\langle\psi| V^n |\phi\rangle$ for $|\psi\rangle, |\phi\rangle$ two orthogonal states in the GS manifold. Recall that mapping between orthogonal states in the GS manifold requires flipping a loop of spins that winds around one dimension of the torus (see Fig. 3), therefore the first non-zero correction will not show up until the L-th order, where L is the smallest circumference of the torus, and the corrections vanish in the thermodynamic limit [1].

We can take advantage of this protected ground state for quantum computation. From here on we will use the terms spin and qubit interchangeably, but this is just a matter of terminology. A *qubit* can be any 2-level quantum system; referring to it as such simply indicates our intention to use this system for information processing. The Hilbert space of n qubits has dimension 2^n, so we can store 2 qubits worth of information in the ground state of the toric code. The properties of our Hamiltonian ensure that this information is distributed across the state of the entire system rather than localized to the states of individual spins; this protects the information from local errors. A scheme for storing quantum information in this way is called a *quantum error-correcting code* [4, 5], which is where the name toric code comes from [1].

It turns out we can do more than just store information. Let us re-label the four orthogonal ground states (a, b, c, d) from Fig. 3 as $|00\rangle, |10\rangle, |01\rangle$ and $|11\rangle$, to make the analogy explicit for two *logical qubits* (as opposed to *physical qubits*). Then a string of X operators looped around the horizontal dimension maps $|00\rangle \leftrightarrow |10\rangle$, and a loop of Z operators in the vertical direction maps $|00\rangle \leftrightarrow |00\rangle$ and $|10\rangle \leftrightarrow -|10\rangle$, therefore we can identify these strings as *logical Pauli operators* $\bar{X}_1$ and $\bar{Z}_1$ acting on the first logical qubit. Similarly, the remaining loops of X and Z operators act as logical operators $\bar{X}_2, \bar{Z}_2$ on the second qubit (Fig. 6).

Note that the action of these logical operators does not depend on a specific path as long as that path winds around the torus, and we could also interpret these operators as anyon pairs which tunnel around the torus before self-annihilating. This is just one example of the connections between topological order, anyons, and quantum computation. A further discussion is beyond the scope of this review, but before we conclude it should be pointed out that the toric code and its close relative the *surface code* [6] underpin some of the leading proposals for fault-tolerant quantum computation [7]. This section is intended to provide some insight into why that is the case, and to encourage further reading (e.g. ref. [5] is quite accessible).

V. CONCLUSION

The toric code is a canonical example of a system with topological order and anyonic excitations. It also establishes a connection between physical systems and quantum information, which motivates the study of anyons and topological order as a route to fault-tolerant quantum computation. The quest to better understand systems where topology plays a role and to fabricate them in the lab is an active area of research, and there is much to discover in the coming years.

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