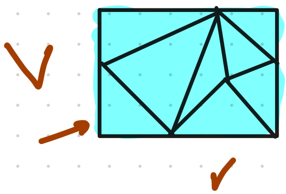
Topology and insulators

LECTURE 17

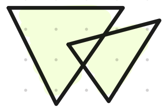
The Euler characteristic of a square

There is a rule how to partition a square into Δ pieces

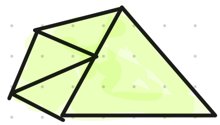


The rule says the pieces must fit together along the edges

Bad examples:



no overlaping allowed



the vertex of a Δ cannot touch the edge of another Δ

Let's count the following elements

Δ faces	f	8
edges	e	16
verteces	v	9

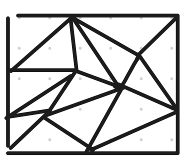
Now we calculate $f - e + v = 8 - 16 + 9 = 1$

The partioning of a figure K into Δs following the above rule is called a triangulation of K.

The number $\underbrace{f - e + v}$ is called the Euler characteristic $\chi(K)$:

$$\boxed{\chi(K) = f - e + v}$$

Problem: Calculate Euler characteristic of

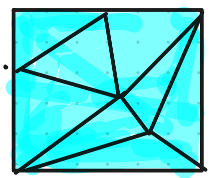


So now you noticed does not matter how you triangulate the answer for $f(\kappa)$ is the same.

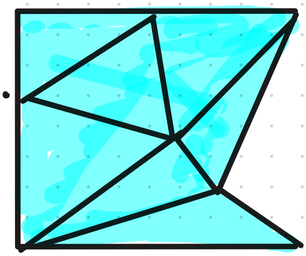
But what if we Δate the a squre into a billion Δs? We cannot simply count it.

Example: Theorem $\underbrace{f(\square)=1}$

Consider any Δation

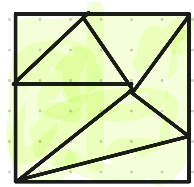


remove 1 Δ

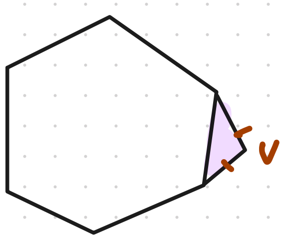


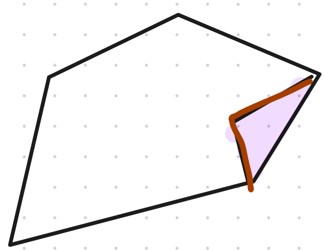
deform it

$\Downarrow$



iF you observe a process of deformation You will find that we always do it by two ways:





here f, e, v change

$f \to \underline{f+1}$ $e \to \underline{e+2}$ $v \to \underline{v+1}$ ✓

$f-e+v = \text{CONST}$ ✓

$f \to f+\underline{1}$ $e \to e+\underline{1}$ $v \to v$

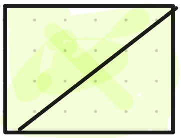
$f-e+v = \text{CONST}$ ✓

But in both cases

$f - e + v$ is constant!

B/c any Δation of a square is obtained as follows:

$f = 2$
$e = 5$
$v = 4$



$f - e + v = 1$

We add some triangular as above and deform the resulting into a square : $\Rightarrow$ any triangulation has $f - e + v = 1$. EOP

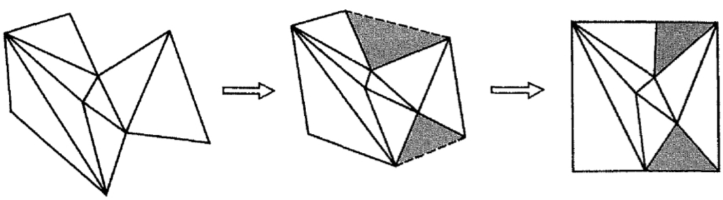
(btw we proved it by induction) starting with $f = 2$

Once we know $\chi(\square) = 1$, we can say that

$\boxed{\chi(\text{polygon}) = 1}$ on the plane

Proof:

After we triangulate a polygon we can add a Δ and deform the resulting figures to a $\square$.



add shaded triangles

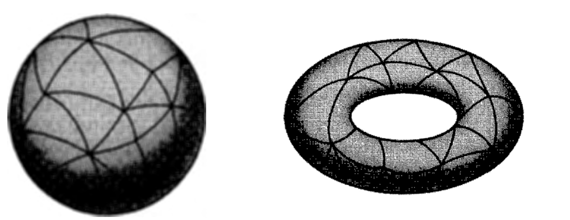
$f \to f + 2$
$e \to e + 2$
$v \to v$

$\chi(\square) = 1$

Next we switch to other surfaces:

Sphere and torus

Now we can make a patch work on the surface of torus or sphere to produce a similar triangulation.

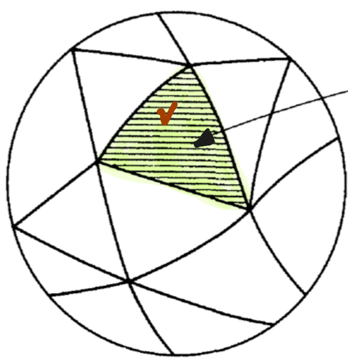


What is χ(sphere) and χ(torus)?

Theorem: $\underbrace{\chi(\text{sphere}) = 2}$ and $\underbrace{\chi(\text{torus}) = 0}$

I only prove it for a sphere.

Consider a Δation of the sphere and let's remov 1 Δ.



Since the sphere with minus one Δ can be stretched to the plane. This figure is equal to the Large triange with χ(polygon) = 1

Notice the number of c and σ remains the same but the number of faces f is changed by 1.

$$\chi(\text{sphere}) = \underbrace{\chi(\text{sphere} - \Delta)}_{\text{polygon} \equiv 1} + 1 = 2$$

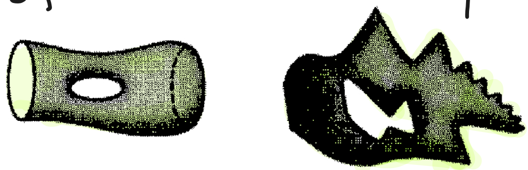
EOP

CLOSED SURFACE

in topology we consider 2 figs to be different if we cannot transform those via elastic deformations.

But what if a figure has holes?

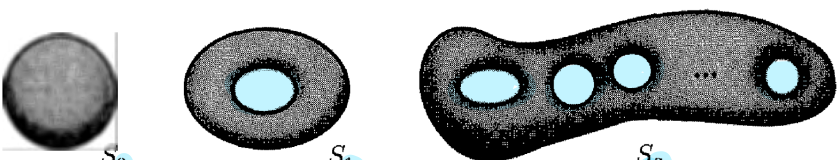
Example of non-closed surfaces:



(a) it has an edge

(b) the surface has singularities

Let's introduce a surface with a hole(s)



S_0 S_1 S_g

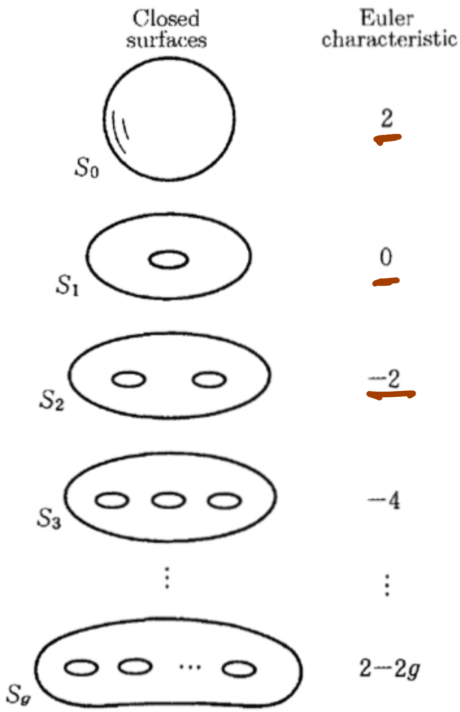
a hole is called genus

Using this notation we can write the Euler characteristics as:

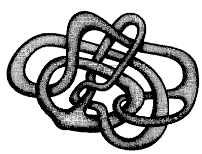
$\chi(S_0) = 2$ $\chi(S_1) = 0$, so what's $\chi(S_g)$?

I'm going to skip the proof but here is the theorem:

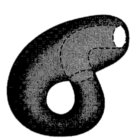
$$\boxed{\chi = 2 - 2g}$$



Fun questions: 1) How many holes or what is g for this Fig.



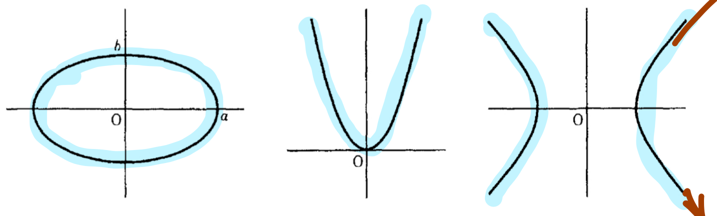
2) What is genus of this figure (Klein bottle)?



CURVATURE OF SURFACE

GAUSSIAN CHARACTERISTICS

Q: Is there any connection between $\chi(S_g)$ and curvature of a surface?

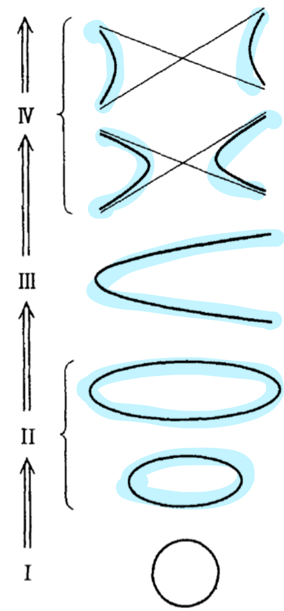


Ellipse Parabola Hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad y = ax^2 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

a and b > 0   a, b > 0

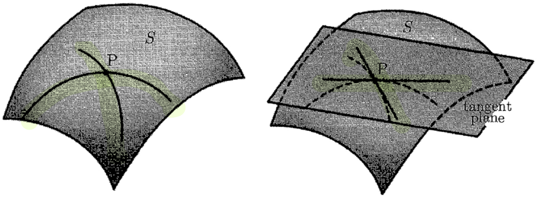
those 3 figures a called conic sections



Flow chart of intersections of a double cone and a plane from various positions

the reason we call it Conical b/c those are cross-sections of a cone by a plane

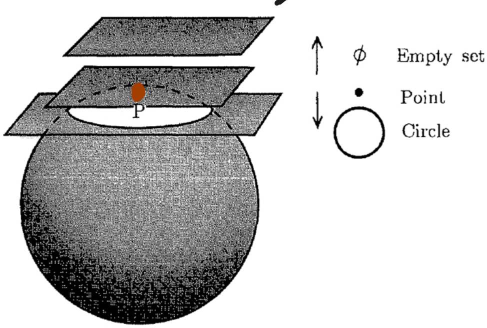
Tangent Plane

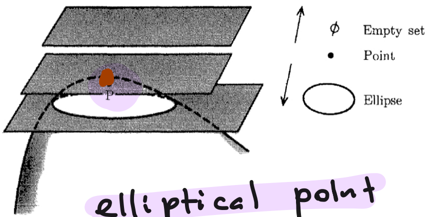


The protocol is simple

- Cut the figure by a plane which is going through the point P
- Then do this again for the second time:
- Draw a tangent line to the obtained cut curves at poin P
- Draw a plane which includes P and 2 tangent lines. This plane is called the tangent plane.

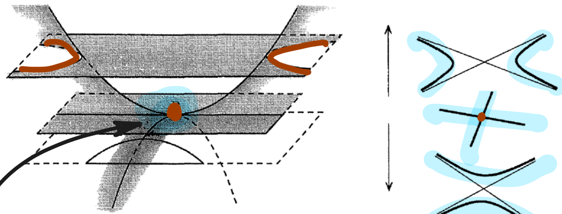
- The way to observe the curvature of the surface S is to shift this plane up and down at P, like this:





elliptical point

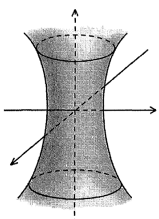
For the convex surface the result is the same.



hyperbolic curving
or hyperbolic point

But if our surface contains a saddle point
The result is very different.

Now we can say that every point on the surface of a sphere is elliptical.



But every point on the surface of a hyperboloid is hyperbolic

Q: Can the nature of a surface point change? ✓
A: Yes!

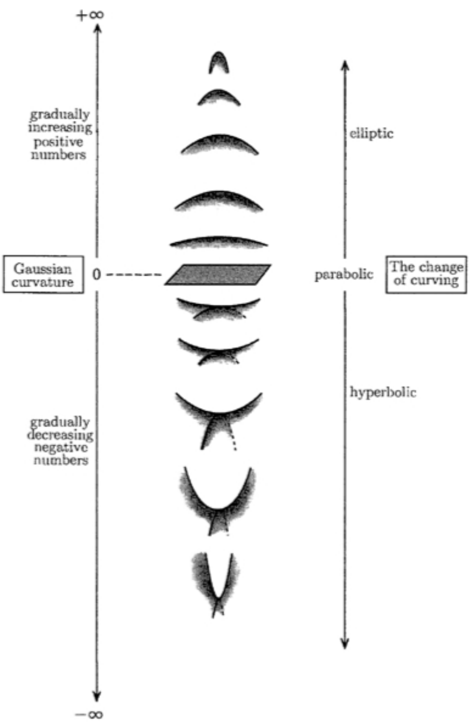
Consider a ball made of clay. Every point on the surface of the ball is an elliptic point. But if we press it with our fingers and make an indentation, then hyperbolic points appear on the indented region of the surface. In this process of changing from convexity to concavity, parabolic curving appears at the moment where convexity changes into concavity. If we move even slightly away from this moment, parabolic curving immediately changes to elliptic or hyperbolic curving. Thus, we can say that parabolic curving is unstable.

Problem: What surface changes its curving in the following way

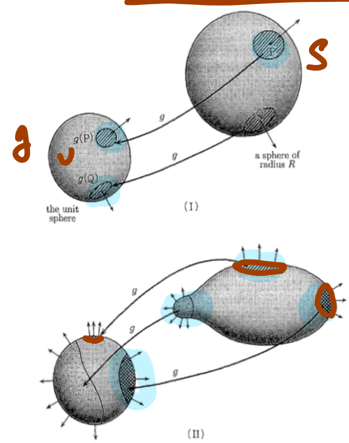
elliptic $\rightarrow$ parabolic $\rightarrow$ hyperbolic?

As you noticed a point on any surface can be characterized by those 3 categories. But can we measure the curvature qualitatively?

Enter the **GAUSSIAN CURVATURE**



The way we are going to calculate the curvature is to map a point P on the surface S to the unit sphere area



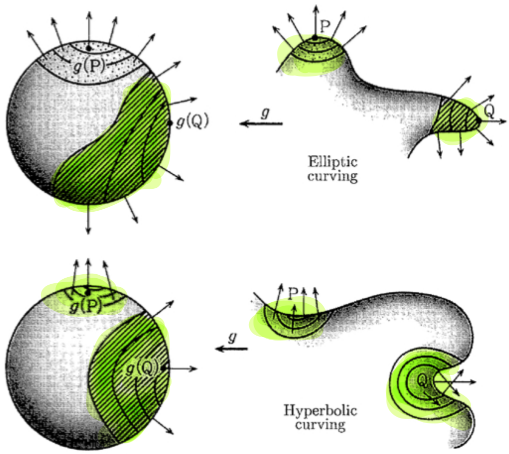
$$K(P) = \lim_{\sigma \to P} \frac{\text{area of } g(\sigma)}{\text{area of } \sigma}$$

Gauss - Bonnet theorem

Let's calculate Gaussian curvature of a sphere, since a sphere has a constant radius R

$$K(P) = 1/R^2$$

▷ To arrive at this conclusion we map the sphere to the unit sphere by the transformation with a similarity ratio $= \frac{1}{R^2}$



Intuitive picture of a gaussian mapping

▷ Imagine that the surface is made of rubber. The mapping of a small region σ by the G-map to the unit sphere is equivalent to cutting out σ off the surface S.

▷ Next we stretch and shrink it to the curving and then "gluing" it into the unit sphere.

▷ Simply, S is a cut into small pieces and then glued into the unit sphere after stretching, shrinking and reversing of each piece.

Gauss - Bonnett theorem

▷ First recall we can deform the closed surface in a concave and convex manner.

▷ Pushing a certain part causes another area to lose its convexity and even become dented inwards.

▷ if the sphere is deformed and some part of it starts having a greater Gaussian curvature, then the curvature of some other parts of the surface will decrease.

Now the theorem itself:

The total sum of the Gaussian curvature $K(P)$ over a surface is equal to the Euler characteristic χ of the surface $\times 2\pi$

$$\frac{1}{2\pi} \cdot \int_S K(P)\, d\sigma = \chi(S)$$

▷ We will not prove the theorem, but let's verify it on a sphere $K(P) = 1/R^2$

$$\frac{1}{2\pi}\int_S K(P)\, d\sigma = \frac{1}{2\pi}\cdot\frac{1}{R^2}\cdot\underbrace{\int_S d\sigma}_{\text{surface of a sphere}} = \frac{1}{2\pi}\cdot\frac{1}{R^2}\cdot 4\pi R^2 = 2\,!$$

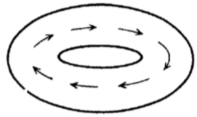
Recall $\chi(S_0) = 2$

▷ The fundamental result:

$$\boxed{\frac{1}{2\pi}\int_S K(P)\, d\sigma = \chi(S_g) = 2-2g}$$

Vector Fields on Surface

Here is the wind blowing on the sphere



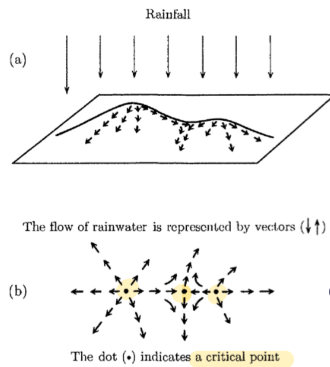
The same on torus

Q: What's the difference?

BTW: if we have a surface described by the parametric $S(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \Rightarrow$ $$\boxed{K = \frac{x'(t)\, y''(t) - y'(t)\, x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}}$$

▷ Imagine that the surface is a field with alternating regions of concavity and convexity and with small vectors in them.

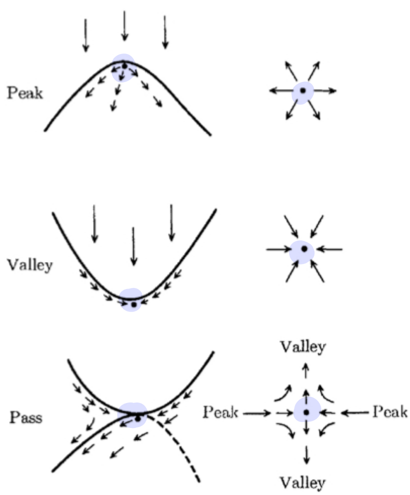
▷ The places where flow stops is called a critical point



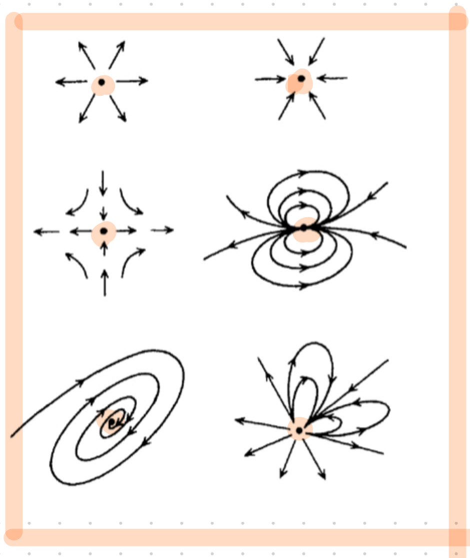
critical points •

▷ Very generally we can think of what critical points are possible?

Check this figure:



▷ If you spend enough time you can descover that the only critical points are:



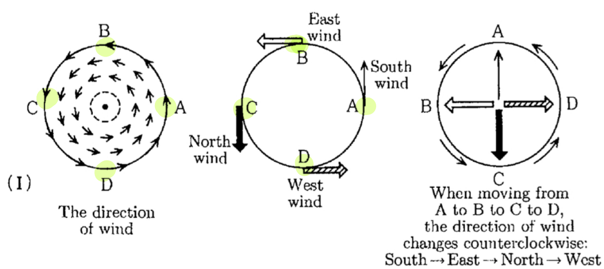
▷ Let's introduce a new tool: the index of a critical point

e.g. if the top of a mountain is as flat as the top of a table, the fallen rain will collect on those flat areas. In such a case the critical points are everywhere.

▷ Now think of a flow of H_2O entering & leaving the critical point.

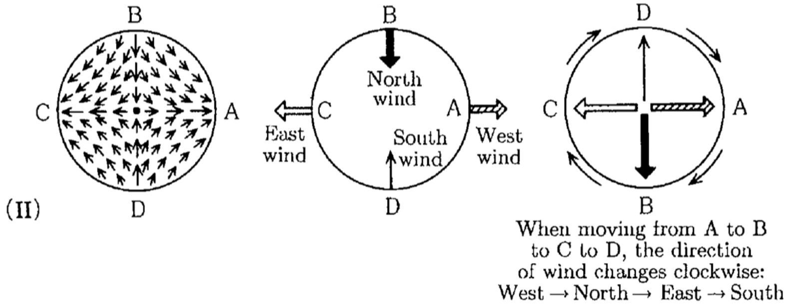
▷ To calculate the index of the critical point draw a closed path around it which contains only one critical point.

▷ Move around the path and measure the direction of the flow at each point



▷ Since the wind directions revolve once around the circle counter-clockwise we assing the index $P = +1$

▷ What about the critical point index here?



$P =$

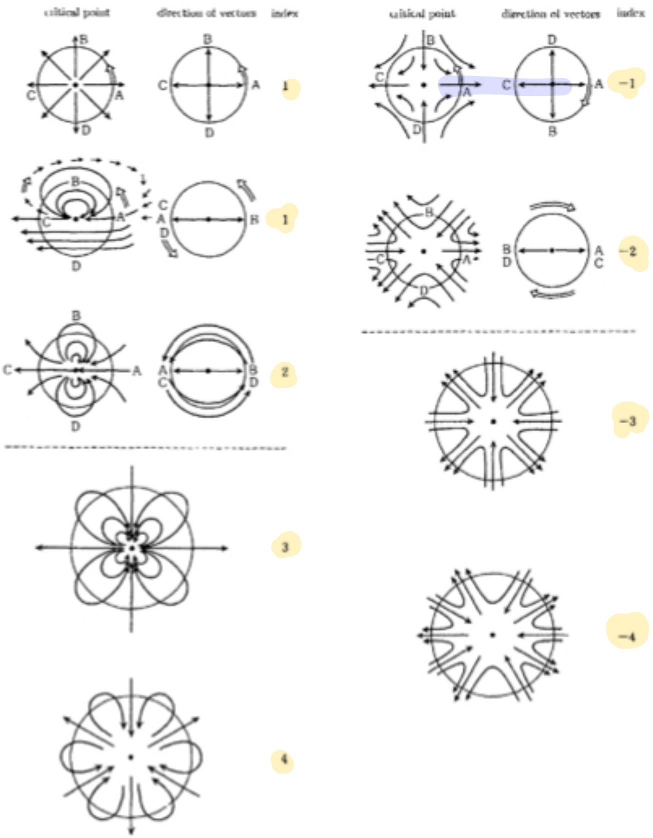
▷ General protocol: Draw a circle around a critical point P, and go around counter clockwize, observe change in the direction of a vector.

If the direction of the vector revolves n times we say $i(P) = +n$

Otherwize $i(P) = -n$

Next page figure shows few interesting examples:

Examples of critial points and their indeces.



Here is the very important theorem

THE POINCARE - HOPF THEOREM

Let S_g be a closed surface. For any vector field on S_g with finitely many critical points, the sum of the critical point indeces $i(P)$ is equal to the Euler characteristic of S_g

$\chi(S_g) = 2 - 2g$

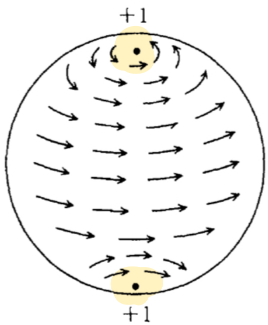
$$\sum i(P) = 2 - 2g$$

▷ Few comments are due:

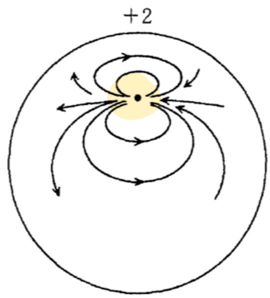
if the surface is a sphere then $\chi(S_0)=2$

What can we say about the surface?

▷ Apply the P-H theorem we can immediately say that somewhere on the Earth there are 2 and only 2 points where there is no wind.



The flow of west winds



▷ Consider another example

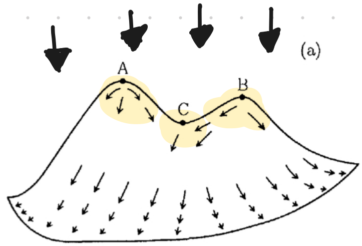
a dipolar magnetic monopole on the surface of the sphere.

The critical index of this point is $i(P)=+2$

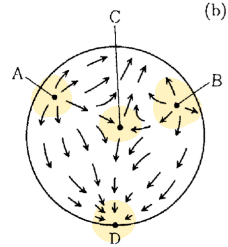
it means that's the only one critical point for this kind of vector field.

▷ Here is another example of a "sphere"

rain fall



The flow of water stops at the base of the mountain



Point D where the base of the mountain has been gathered up

This vector field has 4 critical points

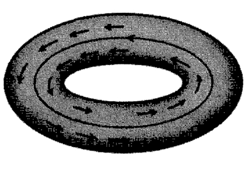
- 2 at peaks A & B $\quad i(A) = i(B) = +1$
- at pass C $\quad i(C) = -1$
- point D where water gathers $\quad i(D) = +1$

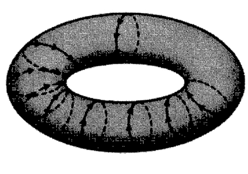
The sum of the indexes $\sum i(A,B,C,D) = +1+1-1+1 = +2$

$\chi(S_0) = 2$ as well.

▷ Now let's consider torus S_1 with $\chi(S_1) = 0$

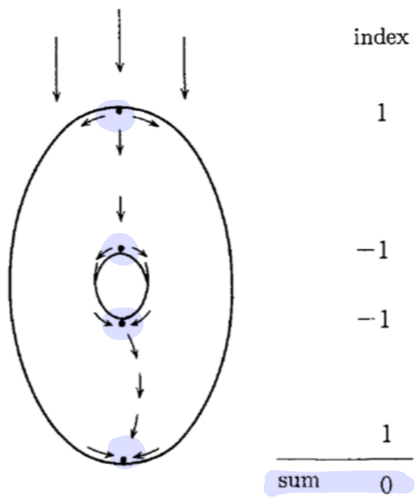
This means that there is a vector field with no critical points or ALL critical point indeces can be only compensated so their sum is zero!





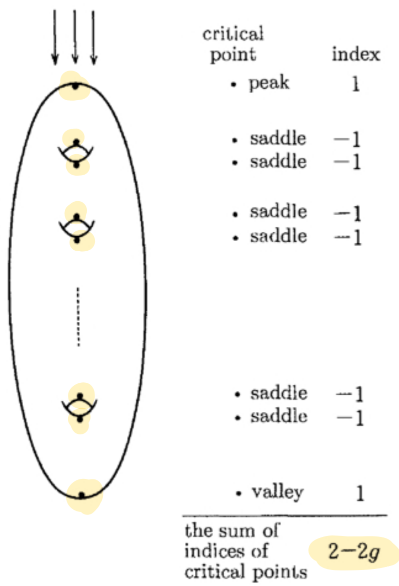
Field with no critical points!

But what if the rainfall vector field drops on the torus



Again we have 4 critical points

Now we can generalize it. to any figure with n-holes:



SUMMARY

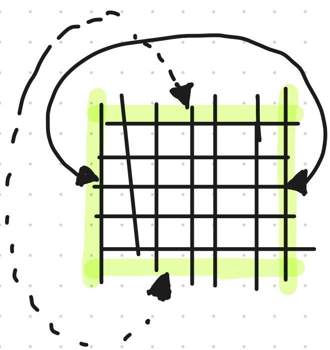
$$\boxed{\chi(S_g) = 2 - 2g = \sum_{S_g} i = \frac{1}{2\pi} \cdot \oint_S K(P)\, d\sigma}$$

Let me now to tell you something about e^- in solid state materials

i = singularity in $E(\bar{k})$ dispersion

$K(P)$ = curvature = Berry potential

g = because of the periodic boundary conditions



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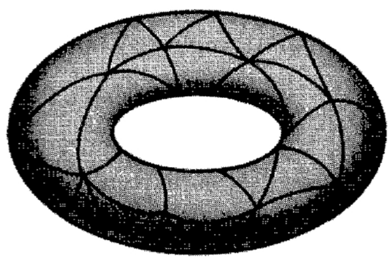
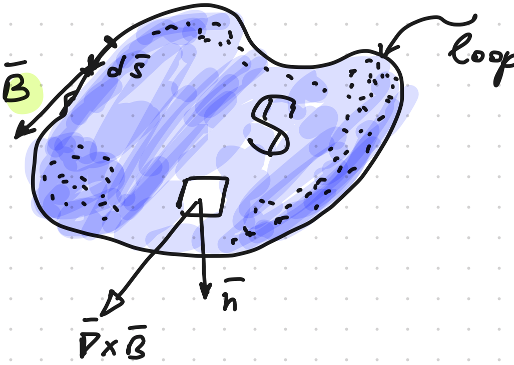


$\chi(S_1) = 0$

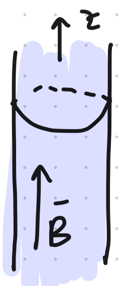
▷ Recall in magnetostatics:

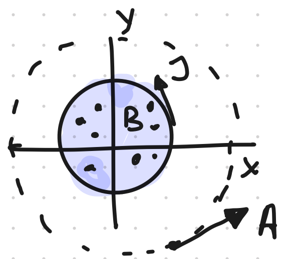
$$\oint \bar{B}\cdot d\bar{s} = \iint \underbrace{(\nabla\times\bar{B})}\cdot\bar{n}\, ds$$



▷ and also we can write

$$B = \bar{\nabla}\times\bar{A}$$





$B_x = B_y = 0$

$B_z \neq 0$

$A_x \neq 0$

$A_y \neq 0$

$$\oint \bar{A}\cdot d\bar{s} = B\cdot\pi r^2 = \text{FLUX}$$

Topology	Magnetism	Quantum
Gaussian curv. K(P)	$\nabla\times B$	Berry curvature $\bar{\nabla}\times\langle u\|\bar{\nabla}_\lambda\|u\rangle$
Total curvature Genus $2-2g$ Σ crit. indeces	FLUX OF B Through S	Berry phase

▷ Berry etc.

$$\oint_\lambda \underbrace{\langle u_\lambda|\nabla_\lambda|u_\lambda\rangle}_{\equiv A(\lambda)\ \equiv\ \text{Berry potential}} d\lambda \equiv \oint \bar{A}(\lambda)\cdot d\bar{\lambda} \equiv \iint_S \underbrace{\bar{\nabla}\times A}_{\equiv\ \text{Berry curvature}}\, ds$$

Berry phase $\varphi = -\mathrm{Im}\,(\text{total Berry curvature})$

also $\oint \bar{A}(\lambda)\, ds$ = FLUX OF BERRY CURVATURE

$\Rightarrow$ Berry phase $= -\mathrm{Im}\left[\text{FLUX OF BERRY CURVATURE}\right]$

$\equiv$ SUM OF CRITICAL INDEXES OF TOPOLOGICAL CHARGES OF

Solid State & Topology

Topological index for an insulator

The Berry curvature and the Chern number.

for an insulator we can use the Bloch waves to define a curvature in the 2D - k space. for Bloch waves

$\psi_{n,k}(\bar{r}) = U_{n,k}(\bar{r})\, e^{i\bar{k}\bar{r}}$, we define the k-space curvature (Berry curvature) as

$$\mathcal{F}_n(k) = \iint_{\text{unit cell}} |\nabla_k U_{n,k}(\bar{r})|^* \times \nabla_k U_{n,k}(\bar{r})\, d\bar{r} =$$

gradien ∇_k defines the vector in the k-space

$$= \epsilon_{ij} \iint_{U.C.} \left|\frac{\partial}{\partial k} U_{n,k}(\bar{r})\right|^* \frac{\partial}{\partial k} U_{n,k}(\bar{r})\, dr$$

ϵ_{ij} = Levi-Civita symbol $\quad \epsilon_{xx} = \epsilon_{yy} = 0$, $\epsilon_{xy} = -\epsilon_{yx} = 1$

FROM the MATH p.o.v the BERRY CURVATURE AND THE GAUSSIAN CURVATURE ARE THE SAME

The total Berry curvature is the topological index!

The topological index is defined as

$$\boxed{C_n = \frac{1}{2\pi} \oiint_{BZ} \underbrace{\mathcal{F}_n(k)}_{\text{Gaussian curvature in BZ}} d\bar{k} \equiv \text{the Chern number}}$$

For each band n, we can define such a number C_n and for an insulator the total Chern #:

$$\boxed{C = \sum_{n \uparrow \text{over the filled bands}} C_n}$$

- The total chern number C, is the same as the number of chiral edge states.
  e.g. if $C=0$ we have a trivial insulator without edge states $\sigma_{xx} = \sigma_{xy} = 0$.
- If $C \neq 0$ we call such an insulator a TI or the Chern insulator.
  This insulator will have the edge states with $\boxed{\sigma_{xx} = 0}$ and $\boxed{\sigma_{xy} \neq 0 = C \frac{e^2}{\hbar}}$ for the Hall conductivity.
- Let me also claim without a proof.
  For a metal or an insulator the Hall conductivity is the total Berry phase curvature summed over all occupied states. For metal we sum up over occupied (valence) and partially occupied bands (conduction bands):

$$\sigma_{xy} = \frac{e^2}{\hbar} \sum_{n,\,\text{valence band}} \left[\frac{1}{2\pi}\int_{BZ} d\bar{k}\, \mathcal{F}_n(\bar{k})\right] + \frac{e^2}{\hbar} \sum_{n,\,\text{conduction}} [\ldots]$$

- Few important points. For the gaussian curvature, the total K is only quantized if the surface has no boundaries

- For the Berry curvature is the same. if we integrate over the whole BZ we will have a quantized Chern number.
- However, if we integrate over a part of BZ the C is non integer.

- For a metal we need to integrate only over the filled states or the Fermi sea and as such there is a boundary set by the Fermi surface. As such C is not quantized. That is why we have no quantized Hall conductivity for metals but we do have this for insulators. (semiconductors)

- So by Chern # we define TI but not Topological metals.

OTHER TOPOLOGICAL INDICES.

If in addition to the C number, we demand a certain symmetry to be present e.g. time-reversal (TR) we can introduce different topo indeces.

If any of these indees are non-zero the insulator is also a TI.

This kind of insulators also called the symmetry-protected insulators (SPTI), with the common properties:

- One of the indees is non-zero.
- The bulk is an insulator, but the edge is a metalic state
- The edge state is different from a simple metal in d-1 dimensions. (e.g. 1/2 of the ordinary metal)
- The edge states may have some quantization effect.
- If the symmetry is broken the edge state disappear.

Note: if we assume no symmetry the only TI is the Chern insulator, which is defined in the even space dimensions, i.e. we can have QHE only in 2D but not in 3D.

Note: for SPTIs they can exis for both 2D & 3D if we preserve TR symm. (e.g. NO MAGNETISM)
In 1D we need a very special symmetry called the chiral symmetry to get a TI.

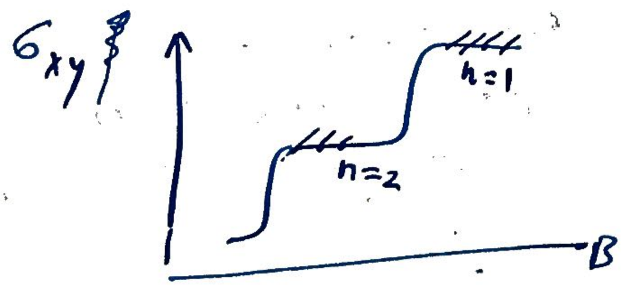
Q. Why TI have metalic states at the edge?

Consider vacuum

vacuum is an insulator (though trivial) with $C = 0$ inside the TI $C \neq 0$.

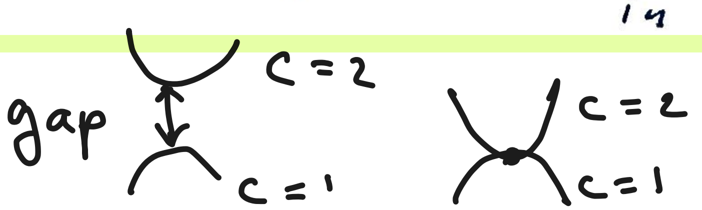
NB! Topology never changes in a smooth way! We cannot deform a sphere into a torus similarly. we cannot transform a trivial or a band insulator into a TI, thus the insulating states need to be destroyed by closing a band gap, or we get a metal.

Q. Why there is a metalic region between two plateaus?



different plateaus have different topological indeces n.
So the story as above to go from $n=1 \to n=2$ need to close a gap to destroy the topology. $\Rightarrow$ metal.

Q Why the Hall conductivity is so exact in a Chern TI?



only possible if the gap $= 0$

Since the Hall conductivity is determined by topology of the wave function it's very robust and precise.

So as long as any perturbation is not changing topology (or destroy protective symmetry) σ_{xy} will be the same for any sample.

In order to do this via some kind of perturbation we need to close a gap first (via doping for example) and only then we can change σ_{xy}.

So technically the error bar in σ_{xy} is 0. (well within how well we know $\hbar$ and e)

Q. So far you talked about non-interacting e^- what if you turn e-e interactions?

For weakly interacting electrons the same connection between topology and Hall (the Berry connection) still remains.

No proof here.

Why is it interesting at all?