Advanced Quantum Mechanics
MAKING THINGS RELATIVISTIC.

Background information:

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi \qquad H = \frac{p^2}{2m} + V(r)$$

non-relativistic version.

in special relativity we can introduce 4-vector

$$p^\mu \cdot p_\mu = m^2c^2 = g_{\mu\nu}p^\mu p^\nu = [E/c \;\; p^1 \;\; p^2 \;\; p^3] \begin{bmatrix} E/c \\ -p_1 \\ -p^2 \\ -p^3 \end{bmatrix} \rightarrow \frac{E^2}{c^2} = p^2 + m^2c^2$$

Lets change dynamical variables to operators: $E \to H \quad p_i \to i\hbar\frac{\partial}{\partial x_i}$

$$H = \sqrt{-\hbar^2c^2\partial_\mu\partial_\mu + m^2c^4}$$

bad news we need a linear theory.

Spin type and personal eqn.

π-meson, Higgs boson = scalars $S=0$
Klein-Gordon eqn.

$S=1/2$ fermions = Dirac eqn. electrons, quarks leptons = spinors

$S=1$ photons and W and Z bosons
= vector particles = Schrödinger eqn.
discovered by A. Proca

Deducing scalar eqn. = Klein-Gordon.

$$\left(i\hbar\frac{\partial}{\partial t}\right)\left(i\hbar\frac{\partial}{\partial t}\right)\Psi = H^2\Psi = (p^2c^2 + m^2c^4)\Psi$$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\Psi = \left(-\hbar^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + m^2c^2\right)\Psi$$

$$-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_0}\Psi - \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + \underbrace{\frac{m^2c^2}{\hbar^2}}_{=\mu^2}\right)\Psi$$

~~SPIN DEPENDENT SCATTERING~~

From this we write down the famous Klein-Gordon equation:

$$\left\{ \frac{\partial}{\partial x^\mu} \cdot \frac{\partial}{\partial x_\mu} + \mu^2 \right\} \psi = 0 \quad \text{or} \quad \left[\partial_\mu \partial^\mu + \mu^2 \right] \psi = 0$$

$$\mu^2 = \frac{m^2 c^2}{\hbar^2} \text{ (units of mass)}$$

$\partial_\mu \partial^\mu = \partial^\mu \partial_\mu \equiv \Box$ = d'Alemberian operator and in 4D Minkowski $\equiv$ 3D laplacian operator $\partial_i \partial_i = \partial^i \partial^i$

This equation specifically describes scalars. $S=0$ particles.

The SOLUTION OF KLEIN-GORDON EQN.

The solution set for the KG equation can be written as

$$\psi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2 V E_n/\hbar}} \Big(A_n e^{-i/\hbar (E_n t - p_n x)} + \underbrace{B_n^\dagger e^{i/\hbar (E_n t - p_n x)}}_{\text{absent in NRQM} \leftarrow} \Big)$$

The descrete set is typical for the system confined into an arbitrary box V.

To rewrite the solution in terms of particles:

$k_i = 2\pi/\lambda_i$ and $p^i = \hbar k^i$ then:

$$p_\mu = \begin{bmatrix} E/c \\ p_i \end{bmatrix} = \begin{bmatrix} E/c \\ -p^i \end{bmatrix} = \hbar k_\mu = \begin{bmatrix} \hbar \omega / c \\ -\hbar k_i \end{bmatrix} \rightarrow$$

$$p_\mu = \begin{bmatrix} E \\ -p^i \end{bmatrix} = k_\mu = \begin{bmatrix} \omega \\ -k^i \end{bmatrix}$$

and if we recall the notation for

$$px = p_\mu x^\mu = Et - p^i x^i = Et - \bar{p} \cdot \bar{x} = p^\mu x_\mu$$

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in Natural units — $E = \omega$, $p_i = k_i$ & $p_\mu = k_\mu$
$p \cdot x = k x$

We can now rewrite our solution $\psi(x)$ in terms of natural units; and in doing this we can turn off the dummy index n, which simply represents an individual wave in the summation.

For a free particle for a given $p = k$ we can label the mode in terms of E_k and ω_k.

$$\psi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left(A_k e^{-ikx} + B_k^+ e^{ikx} \right)$$

Definition of eigen solution:
Each eigen state has a form:

$$\psi_{k,A} = \frac{e^{-ikx}}{\sqrt{V}} \qquad \psi_{k,B^+} = \frac{e^{ikx}}{\sqrt{V}}$$

$$\int \psi_{k,A}^+ \psi_{k,A} \, dV = \frac{1}{V} \int e^{ikx} e^{-ikx} d^3x \equiv 1 \quad \text{or}$$

more generally:

$$\int \psi_{k',A}^+ \psi_{k,A} \, dV = \delta_{kk'}$$, the same for ψ_{k,B^+}

Probability Density of KG eqn

Lets 1st recall whats the probability density for ~~the~~ Sch. eqn. (NRQM)

if we can introduce ψ, then we can calculate $\rho = \psi^* \psi$

Recall the continuity eqn. $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \rightarrow$
implies $\int_V \rho \, dV = \text{CONSTANT IN TIME}$

Detour.

j in NRQM:

$$\psi^{+} \times \left\{ \frac{\partial}{\partial t}\psi = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2M}\nabla^2 + V\right)\right\}\psi\}$$

$+$ For a conjugat Schr. equ.

$$\left\{ \frac{\partial}{\partial t}\psi^{+} = \frac{-1}{i\hbar}\left(-\frac{\hbar^2}{2M}\nabla^2 + \underset{=V}{V^{+}}\right)\psi^{+}\right\}\psi$$

$$\psi^{+}\frac{\partial\psi}{\partial t} + \frac{\partial\psi^{+}}{\partial t}\psi = \psi^{+}\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2M}\nabla^2 + V\right)\psi + \left(-\frac{1}{i\hbar}\left(\left(-\frac{\hbar^2}{2M}\nabla^2\psi^{+} + V^{+}\psi^{+}\right)\right)\right)\psi$$

or

$$\frac{\partial(\psi^{+}\psi)}{\partial t} = -\frac{\hbar}{2iM}\underbrace{\left(\psi^{+}(\nabla^2\psi) - (\nabla^2\psi^{+})\psi\right)}_{\nabla\cdot\left[\psi^{+}(\nabla\psi) - (\nabla\psi^{+})\psi\right]} +$$

$$+ \underbrace{\frac{\psi^{+}V\psi}{i\hbar} - \frac{V\psi^{+}\psi}{i\hbar}}_{=0 \quad V=V^{+}}$$

lets call $\rho \equiv \psi^{+}\psi$

and $j = \frac{\hbar}{2iM}\{\psi^{+}(\nabla\psi) - (\nabla\psi^{+})\psi\} \Rightarrow$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot j = 0$$

For KG equation:

$$\left\{\frac{\partial^2}{\partial t^2}\psi = (\nabla^2 - \mu^2)\psi\right\}\psi^{+}$$

$$-\left\{\frac{\partial^2}{\partial t^2}\psi^{+} = (\nabla^2 - \mu^2)\psi^{+}\right\}\psi \Rightarrow \mu^2\psi^{+}\psi - \mu^2\psi\psi^{+} = 0$$

LHS: $\frac{\partial^2\psi}{\partial t^2}\psi^{+} - \frac{\partial^2\psi^{+}}{\partial t^2}\cdot\psi + \underbrace{\frac{\partial\psi}{\partial t}\frac{\partial\psi^{+}}{\partial t} - \frac{\partial\psi^{+}}{\partial t}\frac{\partial\psi}{\partial t}}_{=0} =$

$$= \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial\psi}{\partial t}\psi^{+} - \frac{\partial\psi^{+}}{\partial t}\psi\right)}_{\text{new LHS}}$$

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RHS before:

$$(\nabla^2\psi)\psi^+ - (\nabla^2\psi^+)\psi + \underbrace{\nabla\psi\cdot\nabla\psi - \nabla\psi^+\cdot\nabla\psi}_{=0} =$$

$$= \underbrace{\nabla\cdot\left(\left((\nabla\psi)\psi^+ - (\nabla\psi^+)\psi\right)\right)}_{\text{new RHS}}$$

So we get:

$$i\times \Big| \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial\psi}{\partial t}\psi^+ - \frac{\partial\psi^+}{\partial t}\psi\right)}_{\rho\ =\ \text{probability density}} = \nabla\cdot\big(\underbrace{(\nabla\psi)\psi^+ - (\nabla\psi^+)\psi}_{j\ \text{probability current}}\big)\big) \rightarrow$$

$$\begin{cases} \rho^{RQ} = j^0 = i\left(\frac{\partial\psi}{\partial t}\psi^+ - \frac{\partial\psi^+}{\partial t}\psi\right) & \text{and} \\ \vec{j}^{RQ} = -i\left((\nabla\psi)\psi^+ - (\nabla\psi^+)\psi\right) \end{cases}$$

~~Recall~~ Recall for NRQM:

$$\left\{ \rho^{NR} = \psi^*\psi \quad \text{and} \quad \vec{j}^{NR} = \frac{\hbar}{2Mi}\left\{\psi^+(\nabla\psi) - (\nabla\psi^+)\psi\right\} \right.$$

are very different!

very strange NRQM and RQM

Lets introduce 4D notation:

$$j^\mu = \begin{bmatrix}\rho \\ \vec{j}\end{bmatrix} = \begin{Bmatrix} j^0 \\ j^i \end{Bmatrix} \Rightarrow \boxed{\frac{\partial j^\mu}{\partial x^\mu} = \partial_\mu j^\mu = 0}.$$

In 4D notation: the 4 divergence of the 4-current of any conserved quantity (total probability) is zero.

What about probability of KG. eq.?

For simplicity lets assume that in the $\psi(x)$ we have only A_k terms and no $B_k^\dagger$

$$\rho = j^0 = i\left(\frac{\partial\psi}{\partial t}\psi^\dagger - \frac{\partial\psi^\dagger}{\partial t}\psi\right) =$$

$$= \left(\sum_k \frac{\omega_k A_k}{\sqrt{2\omega_k}} \frac{e^{ikx}}{\sqrt{V}}\right)\left(\sum_{k'} \frac{A_{k'}^\dagger e^{ik'x}}{\sqrt{2\omega_{k'}}\sqrt{V}}\right) +$$

$$+ \left(\sum_{k'} \frac{\omega_{k'} A_{k'}^\dagger e^{ik'x}}{\sqrt{2\omega_{k'}} \cdot \sqrt{V}}\right)\left(\sum_k \frac{A_k}{\sqrt{2\omega_k}} \frac{e^{-ikx}}{\sqrt{V}}\right)$$

if we integrate $\int \rho \, dv = 1$

$$= \frac{1}{2V}\left(\quad\right)\left(\quad\right) + \left(\quad\right)\left(\quad\right)$$

unless $k \to k'$ $\omega_{k'} \to \omega_k$ the integral $= 0$

so we have in RQM:

RQM:

$$\int \overset{\rho^{NRQ}}{\underset{\neq \rho^{RQM}}{\psi^\dagger \psi}} \, dv = \sum_k \frac{(A_k)^2}{2\omega_k} \neq 1 \quad \text{but}$$

$$\int \underbrace{i\left(\frac{\partial\psi}{\partial t}\psi^\dagger - \frac{\partial\psi^\dagger}{\partial t}\psi\right)}_{\rho} dv = \sum |A_k|^2 = 1$$

NORMALIZATION FACTORS

The total probability is scaler $\Rightarrow$ must be an invariant. So ρ^{RQM} is good.

Now here is the problem.

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Negative probabilities in RQM

~~Proof~~ Prove that: if a particle ψ contains only eigenstates in the form

$$e^{+i(E_n t - \bar{p}_n \cdot \bar{x})/\hbar} = e^{ikX} \xrightarrow{\text{then}} $$ (those are terms with B_k^+)

the total probability is -1.

So the extra states in RQM have physically negative probabilities!

$$\rho = j^0 = i\left(\frac{\partial\psi}{\partial t}\psi^+ - \frac{\partial\psi^+}{\partial t}\psi\right)$$

lets try the simplest solution:

$$\psi = A e^{-ipx} \qquad \psi^+ = A^+ e^{ipx} \qquad px = -iEt + i\bar{p}\bar{x}$$

$$= i\left(A(-iE)e^{-ipx} \cdot A^+ e^{+ipx}\right) -$$

$$- i\left(A A^+ (+i) e^{ipx} e^{-ipx}\right) =$$

$$= 2|A|^2 E$$

E can be positive or negative this in turn means that

ρ can be negative probability density.

This may be very bad, but wait lets see what Feynman and Stueckelberg can tell us.

Interpretation of negative energy states.

N.E.s are particles moving back in time.
$\equiv$ antiparticles

Consider classical electrodynamics:

$$m \frac{d^2 x^\mu}{d\tau^2} = q F^\mu_{\ \nu} \frac{dx^\nu}{d\tau}$$

charge EM tensor

notice $\tau \to -\tau$ is the same as $q \to -q$.

particle moving backword in time =
= antiparticle moving forward in time

Based on this one way to eliminate all the negative states is turn their momentum and charges into opposite and thus generate antiparticles instead, i.e.

$$Et - p\cdot x$$

$t \to -t$ ~~[illegible]~~ means
$E \to -E$

but we need to reverse $p \to -p$

① $E(-t) - p\cdot x$
② $-(Et + px)$
③ $-(Et - px)$

Lets examine the EM current density for KG equation.

$$j^\mu_{em} = q j^\mu$$

where $j^\mu = i\{\psi^+ \nabla\psi - \nabla\psi^+\psi\}$
∂^μ ∂^μ ← as a 4-vector

for $\psi = Ne^{-ipx}$ $\Rightarrow$ $j^\mu_{em} = q\cdot 2|N|^2 p^\mu = q\cdot 2|N|^2(E, \bar{p})$

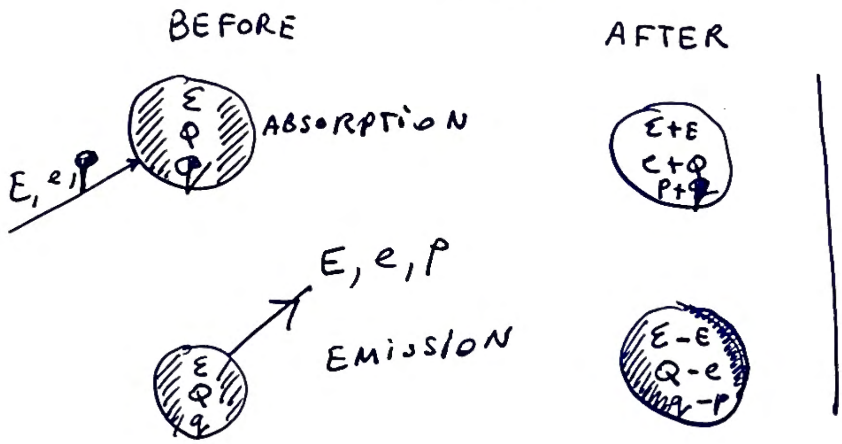
For negative E states $j^\mu_{em} = -q\cdot 2|N|^2(E, \bar{p})$

Recall negative states are bad since they mess up the probabilities, so we want to turn these states into $E>0$:

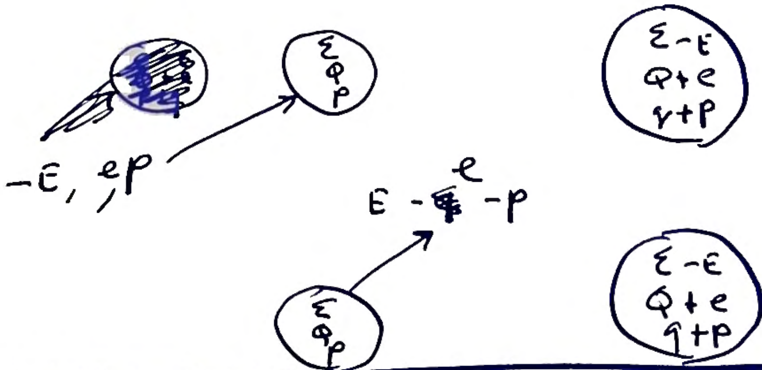
$$\begin{aligned} q &\to -q \\ p &\to -p \end{aligned} \Rightarrow j^{\mu}_{em} = (-q)\, 2|N|^2 (E, -p)$$

or incoming particle is outgoing antiparticles.

Q: Can we apply this protocal to the <u>interacting particles</u>. The anser is yes.



$\Rightarrow$ this is normal



absorption of negative E particles = = emission of negative q and negative e but <u>positive E</u>

Rule:
$E = \pm\sqrt{p^2 + m^2c^4}$ but we accept only $+$
but negative E are antiparticles

$$\Psi(x) = \underset{e^{-i(Et - p\cdot x)}}{\text{incoming } +E} + \underset{e^{+(Et - px)}}{\text{outgoing } +E \text{ antiparticles}}$$