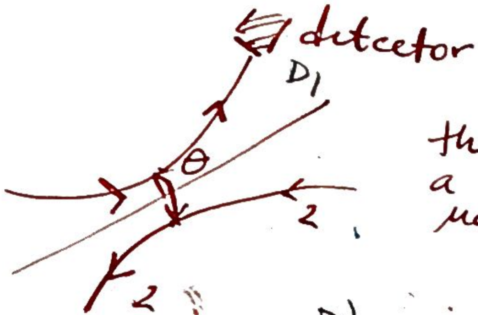
Lecture #6

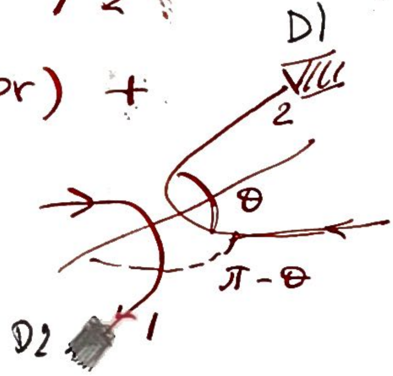
Collission between identical particles

Also read Feynman's Lectures Ch. 4-1



this is a center of mass picture.

(or) +



So, we have two independent channels. And we cannot decide which way we scatter particles 1 and 2, and we have to sum up the amplitudes for both events.

Note, classically we would have a differential cross-section whis is the SUM of σ_i meaning we adding up probabilities and not amplitudes.

$$\frac{d\sigma_{cm}}{d\Omega} = |f(\theta,\phi)|^2 + |f(\underset{\pi-\theta}{\cancel{\theta-\pi}}, \phi+\pi)|^2$$

and f is defined from the scattered w.f.

$$\psi_{r\to+\infty} \to e^{ikr} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

Now recall the product for bosons and fermions is very different.

① Lets assume we scatter 2 spinless bosons

$$r_1 \to r_2 \quad \to \quad |r| \underset{r_1\to r_2}{\overset{\uparrow}{=}} r_1 - r_2 \Rightarrow r \to -r.$$

and in the polar coordinates means that $r, \theta-\pi, \phi+\pi$

2 spinless bosons

$$\psi_s(r \to +\infty) \overset{\Downarrow}{=} e^{ikr} + e^{ikr} + [f(\theta,\phi) + f(\pi-\theta, \pi+\phi)] \frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$$

$$= |f(\theta,\pi)|^2 + |f(\theta, \phi+\pi)|^2 + \underline{2\,\mathrm{Re}\,|f(\theta,\varphi)\cdot f^*(\pi-\theta, \varphi+\pi)|}$$

this is extra compared to classical scattering. = = interference between scattering amplitudes

If the potential is independent of ϕ (e.g. central potential) $\Rightarrow$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 + 2\,\mathrm{Re}\,[f(\theta)\cdot f(\pi-\theta)]$$

Note if $\theta = \frac{\pi}{2}$ $\rightarrow$ $\frac{d\sigma}{d\Omega} = \underline{\underline{4}}\,|f(\theta)|$


$\theta = \frac{\pi}{2}$

↑ the symmetry angle in the c. of. m.

↑ so $1 + 1 = 4$! in quantum mechanics

Moreover recall $f(\theta) \propto \frac{1}{k} \sum_{\ell=0}^{\ell_{max}} (2\ell+1)\, \underline{P_\ell(\cos\theta)}\, e^{i\delta_\ell} \sin\delta_\ell$

to be symmetric $\theta \to \pi - \theta$, it can contain only even ℓs.

② Scattering of 2 fermions spin = 1/2

The total wave. func. must be antisymmetric.

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The <u>spin part</u> of the w.f. can be symmetric ($\uparrow\uparrow$) or antisymm. ($\uparrow\downarrow$), then the spatial part must be symm. for $\uparrow\downarrow$ and antisymm. for $\uparrow\uparrow$. Assume the potential is central and spin independent. $\Rightarrow$

$$\left.\begin{aligned} f_s &= f(\theta) + f(\theta - \pi) \\ f_a &= f(\theta) - f(\theta - \pi) \end{aligned}\right\} \Rightarrow$$

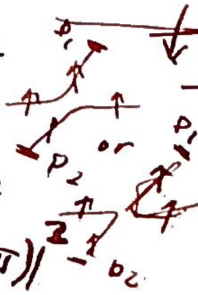
$$\left.\frac{d\sigma}{d\Omega}\right|_{\uparrow\downarrow} = |f(\theta)|^2 + |f(\pi-\theta)|^2 + \underset{=}{2}\,\mathrm{Re}\left[f(\theta) f^*(\pi-\theta)\right]$$

and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\uparrow} = |f(\theta)|^2 + |f(\pi-\theta)|^2 - \underset{=}{2}\,\mathrm{Re}\left[f(\theta) f^*(\pi-\theta)\right]$$

Assume that incoming fermions are up polarized. e.g.

Fraction	S1	S2	Spin in D1	Spin in D2	Prob
1/4	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\uparrow$	$\downarrow$	$\{\uparrow$ / $\downarrow$	$\{\downarrow$ / $\uparrow$	$\lvert f(\theta)\rvert^2$ / $\lvert f(\theta-\pi)\rvert^2$
1/4	$\downarrow$	$\uparrow$	$\{\uparrow$ / $\downarrow$	$\{\downarrow$ / $\uparrow$	$\lvert f(\theta-\pi)\rvert^2$ / $\lvert f(\theta)\rvert^2$



or

or

Total: $= \frac{1}{2}\left[|f(\theta) - f(\pi-\theta)|^2 + \frac{1}{2}|f(\theta)|^2 + \frac{1}{2}|f(\theta-\pi)|^2\right.$

Total cross section =

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - \cancel{2}\frac{1}{2}\, f(\theta)\cdot f^*(\pi-\theta)|$$

Compared to bosons the cross-section is a factor of 4 less.

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Occupation number representation

This idea is very useful for many body theory or condensed mater

1. particle in the box

lets set $\hbar = 1$ $\quad p = -i\frac{\partial}{\partial x}$; $\psi(x) = \frac{1}{\sqrt{L}} e^{ipx}$

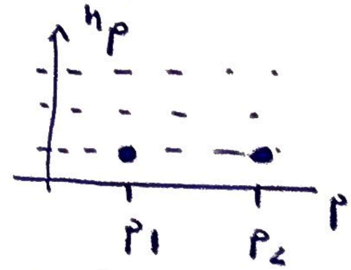
$$p\,\psi(x) = -i\frac{\partial \psi}{\partial x} = p\,\psi(x)$$

if $\psi(x) = \psi(x+L)$ $\quad e^{ipx} = e^{ip(x+L)} \Rightarrow p_m = \frac{2\pi m}{L}$

onto a multi-particle state (e.g. bosons)

$$p\,|p_1 p_2\rangle = (p_1 + p_2)\,|p_1 p_2\rangle$$

$$H\,|p_1 p_2\rangle = (E_1 + E_2)\,|p_1 p_2\rangle$$

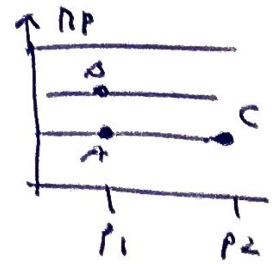


a two particle state $|p_1 p_2\rangle$

What if I have 2 particles in p_3? $\quad E_{p_3} = 2\times E_{p_3}$ double of a single particle energy

$$\sum_m n_{p_m} E_{p_m}$$

n_{p_m} is the total number of particles in the state p_m



In QFT instead of listing what particle is in what state we can say

two particle are in p_1, 1 particle in p_2 etc.

so we just specify how many in what state $p_1 \dots p_N$

eg. $|2100\dots\rangle$

↑ number of particles in this momentum state.

is called <u>occupation number representation</u>

e.g $|q_1 q_1\rangle = |2\,0\rangle$ $\quad |q_2 q_2\rangle = |0\,2\rangle$ $\quad |q_1 q_1 q_1\rangle = |3\,0\rangle$

$|q_1 q_2\rangle = |1\,1\rangle$ $\quad$ etc.

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Occupation number representation

This idea is very useful for many body theory or condensed matter.

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lets set $\hbar = 1$ $\quad p = -i\frac{\partial}{\partial x}$; $\psi(x) = \frac{1}{\sqrt{L}} e^{ipx}$

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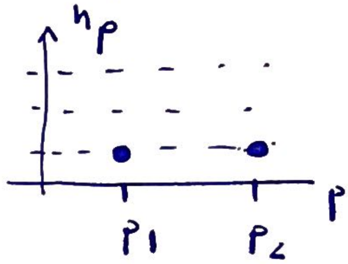
→ onto a multi-particle state (e.g. bosons)

$$p\,|p_1 p_2\rangle = (p_1 + p_2)\,|p_1 p_2\rangle$$

$$H\,|p_1 p_2\rangle = (E_1 + E_2)\,|p_1 p_2\rangle$$



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What if I have 2 particles in p_3? $\quad E_{p_3} = 2 \times E_{p_3}$ double of a single particle energy

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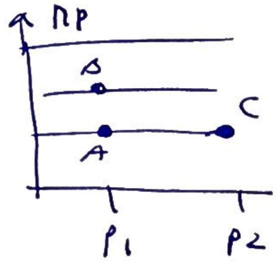

In QFT instead of listing what particle is in what state we can say

two particles are in p_1, 1 particle in p_2 etc.

so we just specify how many in what state $p_1 \dots p_N$

e.g. $|12100\dots\rangle$

↑ number of particles in this momentum state.

is called occupation number representation

e.g. $|q_1 q_1\rangle = |2\,0\rangle$ $\quad |q_2 q_2\rangle = |0\,2\rangle$ $\quad |q_1 q_1 q_1\rangle = |3\,0\rangle$

$|q_1 q_2\rangle = |1\,1\rangle$ $\quad$ etc.

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What happens when we act ~~when~~ on this state by H

$$H\,|n_1\, n_2 \ldots\rangle = \left[\sum_m n_{p_m} E_{p_m}\right] |n_1\, n_2\, n_3 \ldots\rangle$$

simply we find out how many particles in that state × Energy of that state

Big Q: Why do we care?

Recall in harmonic oscillator

$E_n = (n+\frac{1}{2})\hbar\omega$ or $E_n = n\hbar\omega$

So in the oscillat we have n quanta.
and the energy between stats is equallly spaced.

Now imagine N oscillators each labeled by k and the spacing is $\hbar\omega_k$

The total $E = \sum_{k=1}^{N} \hbar\omega_k \cdot n_k$

e.g $k=3$ $\hbar\omega_3$ oscillator has n_3 quanta in it and contributes to the energ $\hbar\omega_3 \cdot n_3$

In general

$$E = \sum_m n_{p_m} E_{p_m}$$

momentum state p_m has n_{p_m} particles in it and contributes $n_{p_m}E_{p_m}$ energy.

So it looks like we can think of a general system as analogous to oscillators.

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What's next: Can we remove the the notion of state vectors at all?

$$|n_1 n_2 \dots\rangle = \prod_k \frac{1}{(n_k!)^{1/2}} (a_k^\dagger)^{n_k} |0\rangle$$

so we retain only one very special state $|0\rangle$

From $|n_1 n_2 \dots n_N\rangle = \frac{1}{\sqrt{n_1! n_2! \dots n_N!}} (a_1^+)^{n_1} (a_2^+)^{n_2} \dots (a_n^+)^{n_N} |0,0,0\dots\rangle$

The general state of the harmonic oscillator:

$$|n_1 \dots n_N\rangle = \prod_k \frac{1}{\dots} \dots$$

e.g. $|121\,000\dots\rangle = \left[\frac{1}{\sqrt{2!}} (a_1^+)^2\right]\left[\frac{1}{1!} a_2^+\right] |0\rangle$

so we can think of this situation as

$a_{p_m}^+$ creates a particle with momentum p_m $|p_m\rangle$

Indistinguishability & SYMMETRY

What I want to do is repeat the same consideration about sym. and antisym. argument for bosons and fermions.

e.g. we have p_1 and p_2 to describe the occupation number $|n_1 n_2\rangle$

$a_{p_1}^+ |0\rangle = |10\rangle \qquad a_{p_2}^+ |0\rangle = |01\rangle$

Lets add another particle in the vacuum:

$$a_{p_2}^+ a_{p_1}^+ |0\rangle \propto |11\rangle$$

$$a_{p_1}^+ a_{p_2}^+ |0\rangle \propto |11\rangle \Rightarrow$$

$$a_{p_1}^+ a_{p_2}^+ = \lambda\, a_{p_2}^+ a_{p_1}^+ \Rightarrow \lambda = \pm 1$$

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$\lambda = 1$ = bosons

$$a^\dagger_{p_2} a^\dagger_{p_1} = a^\dagger_{p_1} a^\dagger_{p_2} \to [\;] = 0$$

$$[a_i \, a^\dagger_j] = \delta_{ij}$$

those commutation rules are the same as for oscillators.

The many particle state of bosons:

$$|n_1 \; n_2 \; \dots\rangle = \prod_m \frac{1}{(n_{p_m}!)^{1/2}} (a^\dagger_{p_m})^{n_{p_m}} |0\rangle$$

$$a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle = a^\dagger_{p_2} a^\dagger_{p_1} |0\rangle = |1_{p_1} 1_{p_2}\rangle$$

in general
$$a^\dagger_i |n_1 \dots n_i \dots\rangle = \sqrt{n_i+1}\, |n_1 \dots n_i+1 \dots\rangle$$
$$a\, |\dots n_i \dots\rangle = \sqrt{n_i}\, |\dots n_i - 1\rangle$$

Case 2: $\lambda = -1 \Rightarrow$

$$\{c_i^\dagger \, c_j^\dagger\} \equiv c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = 0$$

↑ anticommutator

$$c_i^\dagger |n_1 \dots n_i \dots\rangle = (-1)^{\Sigma_i} \sqrt{1-n_i}\, |n_1 \dots n_i+1 \dots\rangle$$
$$c_i |\dots n_i \dots\rangle = (-1)^{\Sigma_i} \sqrt{n_i}\, |\dots n_i - 1 \dots\rangle$$
$$(-1)^{\Sigma_i} \equiv (-1)^{n_1+n_2+n_3 \dots n_{i-1}}$$

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The continiour limit

1) $\delta_{ij} \longrightarrow \delta^3(p)$

As the size of the system goes up spacing in p goes very much down.

$[a_p\ a_q^\dagger] = \delta^3(p-q)$ and

$$H = \int d^3p\ \varepsilon_p a_p^\dagger a_p$$

e.g. For a single-particle state

$$\langle p | p' \rangle = \langle 0 | a_p\ a_{p'}^\dagger | 0 \rangle$$

$$\langle p|p' \rangle = \langle 0 | [\delta^3(p-p') \pm a_p^\dagger\ a_p] | 0 \rangle =$$

$$= \langle 0 | \delta^3(p-p') | 0 \rangle = \delta^3(p-p')$$

So it works and we can rewrite both operators and states in terms of the number of particles ~~in~~ with momentum p and the very special state $|0\rangle$.