LS part 2

Partial wave analysis

If potential is spherically symmetric and Born approx. is not valid.
ψ can be written as a series and each term is called a partial wave.

Scattered waves

in the spherical coordinates:

$$\begin{cases} \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \\ + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \quad \text{and.} \\ L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \end{cases}$$

$$\to \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbb{L}^2}{\hbar^2 r^2}$$

Recall that $(\nabla^2 + k^2)\psi = U(r)\,\psi(r) = F(r)$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbb{L}^2}{\hbar^2 r^2} + k^2 - U(r)\right]\psi_s = 0$$

Lets separate the solutions:

$$\psi(x) = R_e(r)\,Y_e(\theta,\varphi)$$

Note: Since the initial beam is along z, then ψ will have no φ dependence.

Since we are looking for a linear solution

$$\psi = \sum_{e=0}^{\infty} R_e(r)\,Y_e(\theta)$$

The solution corresponds to a partial wave.

multiply $\frac{1}{R_e Y_e}\cdot r^2$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_e(r)\,Y_e(\theta) - \frac{L^2}{\hbar^2 r^2}R_e(r)\,Y_e(\theta) + [k^2 - U]\,R_e(r)\,Y_e(\theta)\right] = 0$$

$$\Rightarrow \boxed{\frac{1}{R_e}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_e(r) + (k^2 - U)r^2 - \frac{1}{Y_e\hbar^2}\mathbb{L}^2 Y_e = 0}$$

Since the 1st 2 terms are only r dependent and the last one is θ dependent we write:

$$\frac{1}{\hbar^2 Y_\ell} L^2 \cdot Y_\ell = \ell(\ell+1) = \text{const.}$$

This is a "well-known" Legendre diff. eqn. ~~w an~~ for θ. The solution $P_\ell(\cos\theta)$.

Now on to the radial part:

$$\frac{\partial^2}{\partial r^2} R_\ell + \frac{2}{r}\frac{dR_\ell}{dr} + \left[k^2 - U - \frac{\ell(\ell+1)}{r^2}\right] R_\ell = 0$$

Substituting $R_\ell \equiv \frac{\chi_\ell}{r}$ $\quad \frac{dR}{dr} = \frac{d\chi_\ell}{dr}\cdot\frac{1}{r} - \frac{1}{r^2}\chi_\ell$

and also $\frac{d^2R}{dr}$ we get

$$\frac{d^2\chi_\ell}{dr^2} + \left[k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell = 0$$

Since we want to consider the process when $r \to \infty$ we get

$$\frac{d^2\chi_\ell}{dr^2} + k^2\chi_\ell = 0,$$ here we assume that $U(r) \xrightarrow[r\to\infty]{} 0$

usual oscillator eqn:

$$\chi_\ell(r) = c_\ell \sin(kr + \Delta_\ell)$$

this is scattered w[ave] solution

Constants

For the incoming wave $U(r) = 0$ we get

$$\frac{d^2\chi_{\ell,in}}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\chi_{\ell,in} = 0$$

the spherical Basel equatio[n]

The spherical Basel has solution:

$$\chi_{e,in}(r) = \sin(kr - l\pi/2)$$

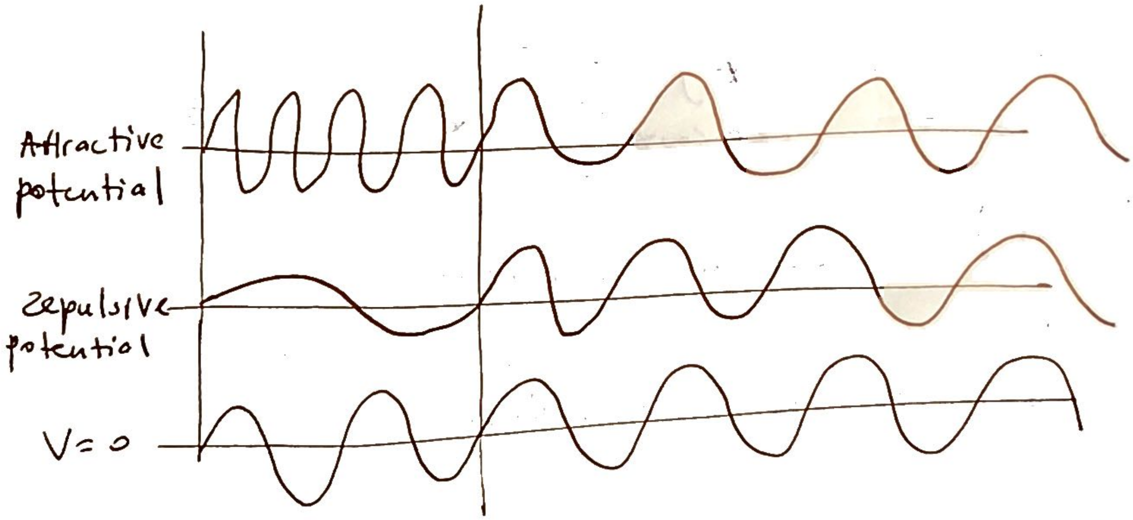
As you noticed the difference between $\chi_{e,in}(r)$ ~~and~~ $\chi_{e,s}$ is in the phase

So what potential does is to shift the wave by δ_e:

$$\boxed{\delta_e \equiv k\cancel{r} + \Delta_e - (\cancel{k}r - \frac{l\pi}{2}) = \Delta_e + \frac{\pi}{2} \cdot l}$$

$l = 0, 1, 2 \ldots$

δ_l is the shift in the l-th partial wave.



Note the amplitud will remain the same for the elastic process.

Phase analysis is used in:

- Bose - Einshtein condensation
- degenerate Fermi gases
- frequency Shifts in atomic clock
- magnetically tuned Feshbach resonances. (Cold atoms)

Scattered amplitudes as ~~...~~ phase shifts

Recall that $\psi = \sum_{l=0}^{\infty} R_l(r)\, Y_l(\theta)$

$\chi_{l,s} = C_l \sin(kr + \Delta_l)$ and $Y_l = P_l(\cos\theta)$

$\chi_l = R_l \cdot r$, and $\Delta_l + \frac{l\pi}{2} = \delta_l$

$\psi = \sum \frac{C_l}{r} P_l \sin(kr + \Delta_l)$, Replace C_l by C_l'/k (we want $k\cdot r$ term) and dropping $'$ we get:

$$\psi = \sum_{l=0}^{\infty} \frac{C_l}{kr} P_l(\cos\theta) \cdot \sin\left(kr - \frac{l\pi}{2} + \delta_l\right)$$

Now recall that very generally we can write down

$$\psi_{r\to\infty} \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

lets express $e^{ikz} = e^{ikr\cdot\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) \cdot j_l(kr)$, we are intereste in the solution for $r \to \infty$ away from the potential.

$$j_l(kr) \underset{r\to\infty}{\Rightarrow} \sin(kr - l\pi/2)/kr$$

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) \cdot \underbrace{\frac{\sin(kr - l\pi/2)}{kr}}$$

Spherical wave with the shift of $l\cdot\pi/2$ and the amplitude which depends on θ.

On the other hand:

Compare ψ and $\psi = \sum \frac{C_l}{kr} P_l \sin\left(kr - \frac{l\pi}{2} + \delta_l\right) =$

$$= \sum_l i^l (2l+1) \frac{P_l}{kr} \cdot \sin\left(kr - \frac{l\pi}{2}\right) + f(\theta)\frac{e^{ikr}}{r},$$

Recall $\sin(x) = \frac{e^{ix} - e^{-ix}}{2} \Rightarrow \frac{\sin(kr - \frac{l\pi}{2})}{kr} = \frac{e^{i(kr - \frac{l\pi}{2})}}{kr} + e^{-i(kr - l\pi/2)}/kr$

$$\sum \frac{c_\ell}{kr} P_\ell \cdot \left[\frac{e^{i(kr - \frac{\ell\pi}{2} + \delta_\ell)} + e^{-i(kr - \frac{\ell\pi}{2} + \delta_\ell)}}{2i}\right]$$

$$= \sum i^\ell (2\ell+1) \frac{P_\ell}{kr}\, \frac{e^{i(kr-\ell\pi/2)} + e^{-i(kr - \frac{\ell\pi}{2})}}{2i}$$

$$+ f \frac{e^{ikr}}{r} \Rightarrow$$

$$e^{ikr}\left[\sum c_\ell P_\ell\, e^{-\frac{i\pi\ell}{2}} e^{i\delta_\ell} - \sum i^\ell(2\ell+1)P_\ell\, e^{-i\ell\pi/2} - i2kf\right] + e^{-ikr}\Big[-\sum c_\ell P_\ell\, e^{i\ell\pi/2} e^{-i\delta_\ell} + \sum i^\ell (2\ell+1) P_\ell\, e^{i\pi/2\cdot\ell}$$

since e^{ikr} and e^{-ikr} are linearly independend whatever is inside $[\ldots] \equiv 0$

$$-\sum c_\ell \cancel{P_\ell}\, \cancel{e^{i\ell\pi/2}}\, e^{-i\delta_\ell} + \sum i^\ell(2\ell+1)\cancel{P_\ell}\,\cancel{e^{i\ell\pi/2}} = 0$$

$$\boxed{c_\ell = i^\ell(2\ell+1)e^{i\delta_\ell}}$$

the same for e^{ikr}

$$f(\theta) = \frac{1}{2ik}\left[\sum c_\ell P_\ell\, i^{-\ell} e^{i\delta_\ell} - \sum(2\ell+1)P_\ell\right]$$

$$= \frac{1}{k}\sum(2\ell+1)P_\ell\, e^{i\delta_\ell}\sin\delta_\ell$$

$$\boxed{f_\ell(\theta) \equiv \frac{1}{k}(2\ell+1)P_\ell\, e^{i\delta_\ell}\sin\delta_\ell}$$

f_ℓ are called the partial wave amplitudes.

We can also see what happens to ψ if we rewrite

$$\psi = \sum \frac{c_\ell}{kr} P_\ell \sin(kr - \ell\pi/2 + \delta_\ell) =$$

$$= \quad c_\ell = i^\ell (2\ell+1) e^{i\delta_\ell} \quad =$$

$$= \sum \frac{i^\ell (2\ell+1) e^{i\delta_\ell}}{kr} P_\ell \sin(k - \ell\pi/2 + \delta_\ell) =$$

$$\updownarrow \quad \frac{e^{ix} - e^{-ix}}{2i}$$

$$= - \sum_{\ell=0}^{\infty} \frac{i^\ell (2\ell+1)}{2ik} \left[\frac{e^{-i(kr - \ell\pi/2)}}{r} - \frac{e^{2i\delta_\ell} e^{i(kr - \ell\pi/2)}}{r} \right] =$$

incoming sph. wave

outgoing sph. wave

$\equiv S_\ell(k)$ ← is called scattering coeff. of the ℓ^{th} partial wave

Here the effect of the potential is in the amplitude of the spherical wave $S_\ell(k)$

Show that: $f_\ell = \frac{1}{2ik}(2\ell+1)(S_\ell(k) - 1)$

$|S_\ell|^2 = 1$ ⇐ conservation of probability.

And finaly for the total cross-section σ:

$$\sigma = 2\pi \int_0^\pi |f(\theta)|^2 \sin\theta \, d\theta =$$

$$= \frac{2\pi}{k^2} \sum_{\ell,\ell'} (2\ell+1)(2\ell'+1) \sin\delta_\ell \sin\delta_{\ell'}^{*} \cdot e^{i\delta_\ell} \cdot e^{i\delta_{\ell'}} \underbrace{\int_0^\pi P_\ell P_{\ell'} \sin\theta \, d\theta}_{= \frac{2\delta_{\ell\ell'}}{(2\ell'+1)}} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell$$

Thus $\boxed{\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell}$

$\sigma = \sum_\ell \sigma_\ell$ if $\delta_\ell = 0$ or π $\sigma_\ell = 0$

and if $\delta_\ell = \pm\pi/2$, $\pm\frac{3\pi}{2}$ it's maxed out.

We can also express σ in terms of f_ℓ

$f = \frac{1}{k}\sum_\ell (2\ell+1) P_\ell\, e^{i\delta_\ell}\sin\delta_\ell$ and for $\theta = 0$

$$\boxed{f_{im}(\theta=0) \underset{\uparrow}{=} \underline{\underline{\frac{1}{k}}}\sum_\ell (2\ell+1)\sin^2\delta_\ell}$$

$e^{i\delta_\ell} = \cos\delta_\ell + i\sin\delta_\ell$

and if we look at $\sigma = \underline{\underline{\frac{4\pi}{k^2}}}\sum_\ell (2\ell+1)\sin^2\delta_\ell$

we see $\boxed{\sigma = \frac{4\pi}{k} f_{im}(\theta=0)}$

Imaginary amplitude of the forward scattering.

This is known as <u>optical theorem</u>!

Lets continue on the same path and determine Relationship among δ_ℓ, $V(r)$ and χ_ℓ.

Recall the equation for χ_ℓ

$$\chi_{\ell,in} \times \Big| \quad \frac{d^2\chi_{\ell,s}}{dr^2} + \left(k^2 - U - \frac{\ell(\ell+1)}{r^2}\right)\chi_{\ell,s} = 0$$

$$\chi_{\ell,s} \times \Big| \quad \frac{d^2\chi_{\ell,in}}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\chi_{\ell,in} = 0$$

and replacing $\chi_\ell = r \cdot R_\ell$

$$\chi_{\ell,in}\frac{d^2\chi_{\ell,s}}{dr^2}+\left(k^2-U-\frac{\ell(\ell+1)}{r^2}\right)\chi_{\ell,in}\chi_{\ell,s}=0$$

$(-)$

$$\chi_{\ell,s}\frac{d^2\chi_{\ell,in}}{dr^2}+\left(k^2-\frac{\ell(\ell+1)}{r^2}\right)\chi_{\ell,s}\cdot\chi_{\ell,in}=0$$

$$\int_0^\infty dr\left|\, rR_{\ell,in}\frac{d^2}{dr^2}(rR_{\ell,s})-rR_{\ell,s}\frac{d^2}{dr^2}(rR_{\ell,in})\right.=$$
$$=r^2U R_{\ell,in}R_{\ell,s}$$

$$rR_{\ell,in}\frac{d}{dr}(rR_{\ell,s})-rR_{\ell,s}\frac{d}{dr}(rR_{\ell,in})\Big|_0^\infty$$
$$=\int_0^\infty r^2\,U R_{\ell,in}R_{\ell,s}\,dr$$

Remember $\begin{cases}\chi_{\ell,in}=\sin(kr-\ell\pi/2) & R_{\ell,in}=\dfrac{\chi_{\ell,in}}{r}\\ \chi_{\ell,s}=i^\ell(2\ell+1)e^{i\delta_\ell}\cdot\sin(kr+\delta_\ell-\ell\pi/2) & R_{\ell,s}=\dfrac{\chi_{\ell,s}}{r}\end{cases}$

$$\boxed{\sin\delta_\ell=-\frac{1}{k}\int r^2U(r)R_{\ell,in}R_{\ell,sc}\,dr}$$

For weak potential $V(r)$ we can use Born approx. when $R_{\ell,s}\approx R_{\ell,in}$ so we get

$$\sin\delta_\ell=-\frac{1}{k}\int r^2\,U(r)\,|R_{\ell,in}|^2dr$$

if the potential is attractive e.g. $\sim -\frac{1}{r}$ $\sin\delta_\ell>0$

if repulsive $\sin\delta_\ell<0$.

Example: For nucleon – nucleon scattering ~300 MeV n-p shift is positive. ⇒ n-p interaction is attractive but ~300 MeV δ_ℓ goes to "0" and becomes <0 so the core is repulsive.

!> Read some interesting discussion on the last paragraph of p. 436 and page 437.

SCATTERING LENGTH

For very low energy only $\ell = 0$ is important

then $f = \frac{1}{k} \Sigma (2\ell+1) P_\ell \, e^{i\delta_\ell} \sin \delta_\ell$

$$\overset{\ell=0}{=} \frac{1}{k} \sum_{\ell=0} (0) \underset{=1}{P_0} \, e^{i\delta_0} \sin \delta_0 \Rightarrow$$

$$\boxed{\begin{aligned} & f(\theta) = \frac{1}{k} e^{i\delta_0} \sin \delta_0 \quad \text{and} \\ & \sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 \end{aligned}}$$

the limiting value of $\underset{E \to 0}{f(\theta)} \equiv -\ell_s$

and $\boxed{\sigma = 4\pi \ell_s^2 \quad \text{and} \quad \delta_0 = -k\ell_s}$

when $k \to 0$