

37. A PHILOSOPHICAL NOTE

NIELS BOHR ONCE SAID THERE ARE TWO KINDS OF TRUTHS. THERE ARE ORDINARY TRUTHS AND THEN THERE ARE DEEP TRUTHS. AN ORDINARY TRUTH IS ONE WHICH THE CONTRARY IS FALSE. EXAMPLE: THE HOUSE IS GREEN AND THE HOUSE IS NOT GREEN. CLEARLY ONE STATEMENT IS FALSE. NOW A DEEP TRUTH IS ONE IN WHICH THE CONTRARY IS ALSO A DEEP TRUTH BUT NOT NECESSARILY FALSE. EXAMPLE: MAN LIVES TO SERVE GOD; OR MAN DOES NOT LIVE TO SERVE GOD. OR PERHAPS, MAN IS NOT A JUST A MACHINE AND MAN IS JUST A MACHINE. THERE ARE NUMEROUS SUCH EXAMPLES OF SUCH DEEP TRUTHS. THINK ABOUT IT —

INTRODUCTION TO PERTURBATION THEORY

WE HAVE BEEN LEARNING THE BASIC MACHINERY OF QUANTUM MECHANICS. WE HAVE TALKED ABOUT THE FREE PARTICLE BEHAVIOR AND HOW WE USE AMPLITUDE WAVE FUNCTIONS TO PREDICT ITS BEHAVIOR. WE STUDIED THE PROPERTIES OF SCATTERING AND QUANTUM ANGULAR MOMENTUM THEORY. RECENTLY WE DISCUSSED THE QUANTUM HARMONIC OSCILLATOR. WE WOULD NOW LIKE TO APPLY ALL THIS POWERFUL MACHINERY TO SOLVING MORE COMPLICATED AND MORE REALISTIC PROBLEM. OUR BASIC PROBLEM WILL BE INVOLVING INTERACTIONS BETWEEN OSCILLATORS AND ELECTROMAGNETIC FIELDS. WE WANT TO WORK OUT SOME OF THE RESULTING ENERGY LEVELS FOR SUCH A SYSTEMS. TO BEGIN WE NEED ONLY A FEW SIMPLE TOOLS; ONE BEING THE SCHRÖDINGER EQUATION FOR THE TIME INDEPENDENT HAMILTONIAN, I.E.,

$$H|\varphi_n\rangle = E_n|\varphi_n\rangle$$

WHERE THE WAVE FUNCTIONS φ_n ARE ORTHOGONAL SUBJECT TO THE CONDITION THAT

$$\langle \varphi_m | \varphi_n \rangle = 0$$

THESE TWO SIMPLE POINTS ARE VERY IMPORTANT TO REMEMBER BECAUSE THEY ARE VERY USEFUL. NOW WE WON'T CONCERN OURSELVES HERE WITH ANY OF THE COMPLICATIONS ARISING FROM A NON-LINEAR OSCILLATOR. IN GENERAL THE NON-LINEARITIES ARE NOT VERY IMPORTANT BUT MORE HONESTLY VERY FEW PEOPLE HAVE SUCCEEDED IN WORKING ANY NON-LINEAR PROBLEMS OUT EXACTLY.

SINCE THE MAJORITY OF QUANTUM MECHANICAL PROBLEMS CANNOT BE SOLVED EXACTLY - EVEN LINEAR PROBLEMS; IT IS OFTEN ADVANTAGEOUS TO APPLY PHYSICAL AND MATHEMATICAL APPROXIMATIONS IN ORDER TO OBTAIN A NEARLY CORRECT ANSWER. THE ABILITY TO WISELY APPLY APPROXIMATIONS AT THE APPROPRIATE TIME IN SOLVING A PROBLEM IS A GREAT GIFT AND ONE WHICH EVERYONE SHOULD ~~WANT~~^{SEEK} TO ACQUIRE. THE GENERAL METHOD OF ATTACKING A PROBLEM IN QUANTUM MECHANIC IS TO APPLY PERTURBATION THEORY.

THERE ARE TWO TYPES OF PERTURBATIONS THEORIES WHICH CAN BE USED. THERE IS THE TIME-INDEPENDENT AND TIME-DEPENDENT PROBLEMS. THE TIME-INDEPENDENT PROBLEMS SEEK TO FIND THE STATIONARY EIGENSTATES AND EIGENVALUES OF THE HAMILTONIAN H . BY A TIME-INDEPENDENT HAMILTONIAN H WE NECESSARILY ARE DEALING WITH AN ISOLATED SYSTEM, ONE WHICH IS NOT INTERACTING WITH AN ADJACENT SYSTEM. SUCH AN ISOLATED SYSTEM CAN UNDERGO A TRANSITION BETWEEN STATIONARY STATES IF THE SYSTEM INITIALLY STARTED OUT IN A STATE WHICH WAS NOT AN EIGENSTATE OF H . MOST SYSTEMS ARE NOT ISOLATED FROM THE EXTERNAL WORLD SO THAT TRANSITIONS BETWEEN DIFFERENT STATIONARY STATES CAN OCCUR WHEN SUBJECT TO EXTERNALLY APPLIED DISTURBANCE. SUCH PROBLEMS AS PARTICLE COLLISION AND DECAY ARE REPRESENTED BY EXTERNAL INTERACTION FORCES. SOLVING THESE PROBLEMS IS OF MORE INTEREST BUT INVOLVES TIME-DEPENDENT ANALYSES WHICH IS INHERENTLY MORE COMPLICATED THAN THE TIME-INDEPENDENT PROBLEMS. WE THEREFORE WILL BE DEALING PRIMARILY WITH THE TIME-DEPENDENT PERTURBATION THEORY.

SEPARABLE SYSTEMS

COMMON TO BOTH TYPES OF PERTURBATION THEORY IS THE IDEA OF SEPARABLE HAMILTONIANS. WHEN WE WRITE THE TOTAL HAMILTONIAN H IN SCHRÖDINGER'S EQUATION

$$-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = H \psi$$

IT IS COMPOSED OF TWO PARTS. THE WAVE FUNCTION ψ IS GIVEN BY $e^{-iE_n t} |\phi_n\rangle$ FOR THE TIME-INDEPENDENT CASE, AND $\sum_n c_n e^{-iE_n t} |\phi_n\rangle$ FOR THE TIME DEPENDENT CASE.

THE HAMILTONIAN H IS EXPRESSED AS

$$H = H_0 + H_1$$

where H_0 IS THE UNPERTURBED HAMILTONIAN WHICH WE POSTULATE WE KNOW HOW TO ACCURATELY DEFINE. H_1 IS THE WEAK INTERACTION HAMILTONIAN WHICH HAS ITS OWN ENERGY SPECTRUM, CALL IT E_{1m} .

THEREFORE THERE ARE TWO SETS OF WAVE FUNCTIONS ψ_0 AND ψ_1 EACH WITH ENERGY LEVELS E_{0m} AND E_{1m} RESPECTIVELY. THE HAMILTONIAN H_0 WILL ONLY OPERATE ON ψ_0 WHILE H_1 WILL ONLY OPERATE ON ψ_1 . THE TWO PIECES OF THE HAMILTONIAN OPERATE ON DIFFERENT PARTS OF THE WAVE FUNCTION. IF WE NOW DEFINE SOME SUPER WAVE FUNCTION Φ WHICH IS A FUNCTION OF THE VARIABLES OF THE SYSTEM THEN

$$H|\Phi\rangle = (H_0 + H_1)|\Phi\rangle = W|\Phi\rangle$$

WHERE W IS THE COMPOSITE ENERGY SPECTRUM OF THE SYSTEM.

JUST AS AN EXAMPLE TO SHOW YOU HOW COMPLICATED H CAN GET LET'S CONSIDER A HELIUM ATOM WITH TWO ELECTRONS AND NUCLEUS. THE COORDINATES OF THE ELECTRONS ARE DENOTED BY (x_1, y_1, z_1) AND THE SECOND ELECTRON BY (x_2, y_2, z_2) . IF WE CONSIDER THE NUCLEUS TO BE OF FINITE MASS THEN IT WILL MOVE AND HAVE COORDINATES (x_N, y_N, z_N) . THEREFORE THE HAMILTONIAN IS

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_N^2}{2M_N} + \frac{e^2}{r_{12}} - \frac{Ze^2}{r_{1N}} - \frac{Ze^2}{r_{2N}}$$

SCHRODINGER'S EQUATION BECOMES

$$-\frac{\hbar^2}{2m} \nabla_1^2 \Phi - \frac{\hbar^2}{2m} \nabla_2^2 \Phi - \frac{\hbar^2}{2M} \nabla_N^2 + V\Phi = E\Phi(x_1, y_1, z_1, x_2, y_2, z_2, x_N, y_N, z_N)$$

IN THIS HAMILTONIAN THE EXTRA TERM H_1 IS ASSOCIATED WITH THE DYNAMICAL RESPONSE OF THE NUCLEUS TO THE MOTION OF THE ELECTRONS AROUND IT. THEREFORE

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{e^2}{r_{12}} -$$

AND

$$H_1 = \frac{p_N^2}{2M_N} - \frac{Ze^2}{r_{1N}} - \frac{Ze^2}{r_{2N}}$$

IN THE LIMIT AS THE NUCLEAR MASS IS TAKEN TO BE INFINITE THEN $H_1 = 0$ AND WE ARE LEFT WITH JUST H_0 .

THE POINT OF WRITING OUT H FOR THE HELIUM ATOM IS TO SHOW WHEN I WRITE $\Phi(x_0, x_1)$ I AM MAKING A SHORTHAND NOTATION FOR THE INDEPENDENT VARIABLES CONTAINED IN THE SYSTEM.

IN ORDER TO SEE HOW $H|\Phi\rangle$ PRODUCES $W|\Phi\rangle$ AS EXPRESSED EARLIER LET'S TRY A COMPOSITE WAVE FUNCTION OF THE FORM

$$\Phi(x_0, x_1) = \phi_n^0(x_0) \phi_m^1(x_1)$$

WE GUESSED AT THE PRODUCT FORM BECAUSE OF OUR EARLIER WORK IN AMPLITUDES. PLUGGING Φ INTO THE ABOVE EQUATION

$$H|\Phi\rangle = (H_0 + H_1)(|\phi_n^0\rangle |\phi_m^1\rangle) = H_0 \phi_n^0 \phi_m^1 + H_1 \phi_n^0 \phi_m^1 = W \phi_n^0 \phi_m^1$$

SINCE H_0 ONLY OPERATES ON ϕ_n^0 AND H_1 ONLY OPERATES ON ϕ_m^1 , WE HAVE

$$\phi_m^1 H_0 \phi_n^0 + \phi_n^0 H_1 \phi_m^1 = \phi_m^1 E_n^0 \phi_n^0 + \phi_n^0 E_m^1 \phi_m^1$$

E_n^0 AND E_m^1 ARE JUST NUMBERS SO THAT THEY CAN COME OUT AS

$$(E_n^0 + E_m^1) \phi_n^0 \phi_m^1 = W \phi_n^0 \phi_m^1$$

SO THAT

$$W = E_n^0 + E_m^1$$

THEREFORE THE ENERGIES ADD WHILE THE WAVE FUNCTIONS MULTIPLY. THIS IS A VERY IMPORTANT PROPERTIES OF SEPARABLE SYSTEMS WHICH WE WILL USE IN MORE COMPLICATED PROBLEM. THE KEY POINT IS THAT THE HAMILTONIAN AND ENERGY SPECTRUMS ARE ADDITIVE.

TIME-DEPENDENT PERTURBATION THEORY

I'D LIKE TO TALK ABOUT THE TIME-DEPENDENT PERTURBATION THEORY BECAUSE IT HAS ENORMOUS PRACTICAL UTILITY IN QUANTUM MECHANIC. IT CONSTITUTES THE UNDERPINNING OF THE ANALYSIS OF 2 SYSTEMS FAR APART OR WEAKLY COUPLED THROUGH A SLIGHT INTERACTING FIELD. A TYPICAL EXAMPLE IS ONE SYSTEM INITIALLY IN THE GROUND STATE WHILE ANOTHER IS IN SOME EXCITED STATE; AT SOME LATER TIME THE TWO SYSTEMS CAN INTERACT AND TRANSFER ENERGY. THE PROBLEM IS TO FIGURE OUT THE PROBABILITY THAT THE TRANSITION WILL OCCUR. SUCH AN EXAMPLE WOULD BE A CAVITY RESONATOR CONSISTING OF MILLIONS AND MILLIONS OF HARMONIC OSCILLATORS WHICH CAN INTERACT WITH SOME ATOMS IN THE CAVITY TO UNDERGO AN ABSORPTION OF A PHOTON OR PERHAPS, EMIT A PHOTON.

AGAIN WE WILL BE CONCERNED ONLY WITH TIME-INDEPENDENT HAMILTONIANS INITIALLY, I.E., $H_0 \neq H_0(t)$. THIS MEANS THE STATE IS INITIALLY PREPARED BY THE EXPERIMENTER. THERE WILL HOWEVER BE A TIME DEPENDENT PIECE OF THE HAMILTONIAN, CALL IT H' , WHICH ARISES FROM THE TIME VARYING INTERACTING FIELD. THE TOTAL HAMILTONIAN IS THEN

$$H = H_0 + H_1(t)$$

AND THE SCHRÖDINGER EQUATION IS

$$-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = H\psi = [H_0 + H_1(t)] \psi(t)$$

WE WILL ASSUME AT TIME $t=0$ WE KNOW PRECISELY THE STATE OF THE SYSTEM TO BE $\psi(t) = \psi(0)$ AT $t=0$. WE WANT TO KNOW WHAT STATE THE SYSTEM WILL BE IN AT SOME LATER TIME t . NOW H_0 DESCRIBES TWO NONINTERACTING SYSTEMS AND $H_1(t)$ DESCRIBES THEIR MUTUAL INTERACTION.

WE WILL SOLVE THIS ~~PROBLEM~~ PROBLEM BY USING AN UNPEDIAN APPROACH. FIRST WE WILL SOLVE THE EASY PROBLEM, I.E.,

$$-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = H_0 \psi(t)$$

BUT TO SOLVE THIS WE NEED TO KNOW WHAT $\psi(t)$ IS. WELL, LET'S TRY THE SOLUTION

$$\psi(t) = e^{-i/\hbar t H_0} \psi(0)$$

IT'S NOT TOO DIFFICULT TO SHOW THIS IS A SOLUTION BUT WHAT THE HELL DOES IT MEAN $e^{-i/\hbar t H_0}$? WHAT IS AN OPERATOR DOING IN THE EXPONENT? WELL IF WE EXPAND THE EXPONENT IN A POWER SERIES THE OPERATION BECOMES CLEAR SINCE WE KNOW WHAT IT MEANS TO OPERATE WITH H_0 ONCE, TWICE, THREE-TIMES, ETC.

NOW IF $\psi(0)$ IS JUST ϕ_n THEN WOULDN'T YOU EXPECT THAT $e^{-i/\hbar t H_0} \phi_n$ WOULD PRODUCE $e^{-i/\hbar t E_n} \phi_n$? SURE YOU WOULD BECAUSE $H_0 \phi_n = E_n \phi_n$. LET'S SEE IF WE CAN PROVE IT, FIRST WE KNOW THAT

$$e^{-i/\hbar t H_0} = 1 - \frac{i}{\hbar} t H_0 + \frac{1}{2} \left(-\frac{i}{\hbar} t\right)^2 H_0 H_0 + \frac{1}{3!} \left(-\frac{i}{\hbar} t\right)^3 H_0 H_0 H_0 + \dots$$

THEN $e^{-i/\hbar t H_0} \phi_n$ PRODUCES

$$\phi_n - \frac{i}{\hbar} t \underbrace{H_0 \phi_n}_{E_n \phi_n} + (\dots) \underbrace{H_0 (H_0 \phi_n)}_{\underbrace{H_0 E_n \phi_n}_{E_n \cdot E_n \phi_n}} + (\dots) \underbrace{H_0 H_0 (H_0 \phi_n)}_{\underbrace{E_n \phi_n}_{E_n^2 \phi_n}} + \dots = e^{-i/\hbar t E_n} \phi_n$$

Now that we understand what the symbols mean we should prove that $\psi(t) = e^{-i/\hbar t H_0} \psi(0)$ is indeed a solution to Schrödinger's equation - so let's see:

$$-\frac{\hbar}{i} \frac{\partial \psi(t)}{\partial t} = -\frac{\hbar}{i} \frac{\partial}{\partial t} \left[e^{-i/\hbar t H_0} \psi(0) \right] = H_0 \psi(t)$$

Pretty easy huh? But you still might be wondering if what I did was mathematically legitimate so I'll prove it. We have to differentiate the power series expansion for $e^{-i/\hbar t H_0}$

$$\begin{aligned} -\frac{\hbar}{i} \frac{\partial \psi(t)}{\partial t} &= -\frac{\hbar}{i} \left[-\frac{i}{\hbar} H_0 + \left(-\frac{i}{\hbar}\right)^2 t H_0 H_0 + \frac{1}{2!} \left(-\frac{i}{\hbar}\right)^3 t^2 H_0 H_0 H_0 + \dots \right] \psi(0) \\ &= H_0 \left[1 - i/\hbar t H_0 + \frac{1}{2!} \left(-\frac{i}{\hbar} t\right)^2 H_0 H_0^2 + \dots \right] \psi(0) \\ &= H_0 e^{-i/\hbar t H_0} \psi(0) = H_0 \psi(t) \quad \text{Q.E.D.} \end{aligned}$$

The notation here is fairly simple but it is hard to figure out what is going on. Now it is necessary to specify the form of $\psi(0)$ which can be done most simply as a power series,

$$\psi(0) = \sum_n c_n \phi_n$$

where the c_n 's can be found by projecting $\psi(0)$ into each ϕ_n as

$$c_n = \langle \phi_n | \psi(0) \rangle$$

This then specifies the initial state of the system but in general $\psi(t)$ is given by

$$\psi(t) = \sum_n c_n e^{-i/\hbar t H_0} \phi_n = \sum_n c_n e^{-i/\hbar t E_n} \phi_n$$

So far we have just concerned ourselves with the first part of the Hamiltonian, H_0 , we must now consider the time dependent part, $H_1(t)$.

Our problem is to solve the Schrödinger equation with the time dependent part of the Hamiltonian. The solution will be of the general form

$$\psi(t) = \left\{ e^{-i/\hbar \int_0^t [H_0 + H_1(t')] dt'} \right\} \psi(0)$$

But wait we can't prove this solution is correct as we did last time. This happens because in general if A and B are two operators then $e^{(A+B)} \neq e^A e^B$

If A and B are non-commutative, i.e., $(AB - BA) \neq 0$. So we have to go about the proof another way. We won't do it too elegantly so no one get lost in the math.

I AM GOING TO TRY AND SOLVE FOR H_1 SUBJECT TO THE CONDITION THAT IT IS TO BE SMALL. I WILL TRY THE SOLUTION OF THE FORM

$$\psi(t) = e^{-i/\hbar t H_0} \chi(t)$$

WHERE $\chi(t)$ IS A SLIGHTLY PERTURBED WAVE FUNCTION FROM ψ_0 .

SUBSTITUTING THIS DIRECTLY INTO SCHRÖDINGER'S EQUATION

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(t) &= i\hbar \frac{\partial}{\partial t} [e^{-i/\hbar t H_0} \chi(t)] = H_0 \chi(t) e^{-i/\hbar t H_0} + i\hbar e^{-i/\hbar t H_0} \frac{\partial \chi(t)}{\partial t} \\ &= H_0 \psi(t) + i\hbar e^{-i/\hbar t H_0} \frac{\partial \chi(t)}{\partial t} \end{aligned}$$

NOW THIS MUST EQUAL $(H_0 + H_1(t)) \psi(t) = (H_0 + H_1(t)) e^{-i/\hbar t H_0} \chi(t)$

THEREFORE WE HAVE AN EQUATION FOR $H_1(t)$

$$H_1(t) = i\hbar \frac{\partial \chi(t)}{\partial t}$$

OR

$$\chi(t) = e^{-i/\hbar \int_0^t H_1(t') dt'}$$

THE ALTERNATE WAY TO EXPRESS THIS DIFFERENTIAL EQUATION IS

$$i\hbar \frac{\partial \chi(t)}{\partial t} = e^{i/\hbar t H_0} H_1(t) e^{-i/\hbar t H_0} \chi(t)$$

THIS IS ALMOST A SCHRÖDINGER EQUATION FOR $\chi(t)$. IF WE DEFINE A NEW HAMILTONIAN

$$H_1'(t) = e^{i/\hbar t H_0} H_1(t) e^{-i/\hbar t H_0}$$

WE HAVE THE SCHRÖDINGER EQUATION

$$-\frac{\hbar}{i} \frac{\partial \chi(t)}{\partial t} = H_1'(t) \chi(t)$$

THIS IS A VERY USEFUL EQUATION BECAUSE IT TELLS HOW THE DISTURBANCE PROPAGATES WITH TIME. FROM THE ABOVE RESULTS WE CAN CONCLUDE THAT IF ψ SATISFIES

$$i\hbar \frac{\partial \psi}{\partial t} = [H_0 + H_1(t)] \psi(t)$$

THEN

$$\psi'(t) = e^{i/\hbar t H_0} \psi \quad \text{WILL SATISFY,}$$

$$i\hbar \frac{\partial \psi'}{\partial t} = H_1'(t) \psi'$$

TO SEE THIS SUBSTITUTE FOR ψ'

$$i\hbar \frac{\partial}{\partial t} [e^{i/\hbar t H_0} \psi] = e^{i/\hbar t H_0} H_1(t) e^{-i/\hbar t H_0} e^{i/\hbar t H_0} \psi$$

$$i\hbar [H_0 e^{i/\hbar t H_0} \psi + e^{i/\hbar t H_0} \frac{\partial \psi}{\partial t}] = e^{i/\hbar t H_0} H_1(t) \psi$$

$$-e^{i/\hbar t H_0} H_0 \psi + i\hbar e^{i/\hbar t H_0} \frac{\partial \psi}{\partial t} = e^{i/\hbar t H_0} H_1(t) \psi$$

DIVIDE THROUGH BY $e^{i/\hbar t H_0}$ AND $\frac{\partial}{\partial t}$ REARRANGE

$$i\hbar \frac{\partial \psi}{\partial t} = [H_0 + H_1(t)] \psi \quad \text{Q.E.D.}$$

IN ORDER TO FIGURE OUT WHAT $H_1'(t)$ REPRESENTS WE NEED TO WORK OUT MATRIX ELEMENTS; THE MATRIX ELEMENT GIVING US THE TRANSITION PROBABILITIES. IN GENERAL

$$\int \phi_m^* A \phi_n dV = \langle \phi_m^* | A | \phi_n \rangle = A_{nm}$$

IN OUR CASE $A = H_1'(t)$ SO THAT

$$\int \phi_m^* H_1'(t) \phi_n dV = \int \phi_m^* e^{i/\hbar t H_0} H_1(t) e^{-i/\hbar t H_0} \phi_n dV$$

SINCE $e^{-i/\hbar t H_0} \phi_n = e^{-i/\hbar t E_n} \phi_n$ AND $\phi_m^* e^{i/\hbar t H_0} = e^{i/\hbar E_m t} \phi_m^*$

WE HAVE THAT

$$\int = e^{i/\hbar (E_m - E_n)t} \int \phi_m^* H_1(t) \phi_n dV = e^{i/\hbar (E_m - E_n)t} \langle \phi_m | H_1(t) | \phi_n \rangle$$

NOW PHYSICALLY ψ' IS THE TIME DEVELOPMENT OF ψ BUT THE PROPAGATION IS MODULATED BY COMPONENT $e^{i/\hbar (E_m - E_n)t}$ DUE TO THE PRESENCE OF $H_1(t)$ OR A SMALL INTERACTION FORCE. ALTHOUGH $H_1(t)$ IS IN GENERAL WEAK THE PHASE SHIFT COULD BE QUITE RAPID.

OBVIOUSLY IF $H_1(t)$ WERE NEGLIGIBLE THEN THE TWO SYSTEMS WOULD NEVER INTERACT AND THERE WOULDN'T BE ANYTHING OF INTEREST OCCURRING. WHEN $H_1(t)$ IS SMALL BUT NOT NEGLIGIBLE WE HAVE SOME INTERESTING PHYSICS. SUPPOSE AT TIME $t=0$ WE SPECIFY $\psi(t)$ PRECISELY, I.E., WE KNOW $\psi(0)$ AND $\psi'(0)$. WHAT OUR FORMULA SAYS THAT IF WE WANT TO KNOW WHAT TO EXPECT OF THE WAVE FUNCTION AT SOME LATTER TIME $t=T$ THEN ALL WE NEED TO CALCULATE IS

$$\psi(T) = e^{-i/\hbar T H_0} \psi'(T)$$

$\psi'(t)$ IS IN GENERAL A POWER SERIES EXPANSION SUCH AS

$$\psi'(t) = \psi_0'(t) + \psi_1'(t) + \psi_2'(t) + \psi_3'(t) + \dots + \psi_n'(t)$$

WHERE THE ZEROth ORDER TERM $\psi_0'(t)$ WE KNOW EQUALS $\psi'(0)$.

WE ARE USUALLY INTERESTED IN THE FIRST ORDER APPROXIMATION WHICH TELLS US TO WHAT ORDER IN H_0 WE NEED, I.E., FIRST ALSO.

OUR SCHRÖDINGER EQUATION BECOMES

$$i\hbar \frac{\partial}{\partial t} [\psi_0'(t) + \psi_1'(t)] = H_1'(t) [\psi_0'(t) + \psi_1'(t)]$$

OR

$$i\hbar \frac{\partial}{\partial t} [\psi_0'(0) + \psi_1'(t)] = H_1'(t) [\psi_0'(0) + \psi_1'(t)]$$

This equation reduces to

$$i\hbar \frac{\partial}{\partial t} \psi'(t) = H'(t) \psi(0)$$

Which can be integrated for $\psi'(t)$ to get

$$\psi'(t) = -\frac{i}{\hbar} \int_0^t H'(t') dt' \psi(0)$$

Therefore we have found $\psi'(t)$ to first order,

$$\psi'(t) = \psi(0) - \frac{i}{\hbar} \int_0^t H'(t') dt' \psi(0)$$

If we want $\psi'(t)$ at time $t=T$ we simply integrate dt' between 0 and T. It is a little more messy to go to the second order terms and higher but they are obtained in a straightforward manner. For the second order term

$$\psi_2'(t) = \left(-\frac{i}{\hbar}\right)^2 \int_0^T dt \int_0^t dt' H'(t) H'(t') \psi'(0)$$

The third order term is

$$\psi_3'(t) = -\left(-\frac{i}{\hbar}\right)^3 \int_0^T dt \int_0^t dt' \int_0^{t'} dt'' H'(t) H'(t') H'(t'') \psi'(0)$$

As we said earlier for the questions we want to answer the first order term will be sufficient.

So now we want to find what the amplitude will be for the system to end up in some final state, ψ_f , if it is initially prepared in state some initial state ψ_i . How do we calculate this. Well if we have the system describable by a wave function $\psi(t)$ we evaluate it at time $t=T$ at the time of interest and then project this wave function into the final state ψ_f' that we'd like it to be in. Mathematically this is said

$$\langle \psi_f' | \psi'(T) \rangle$$

I should point out this is mathematically equivalent to $\langle \psi_f | \psi(T) \rangle$. To prove that it is

$$\begin{aligned} \langle \psi_f' | \psi'(T) \rangle &= \langle e^{\frac{i}{\hbar} T H_0} \psi_f | e^{\frac{i}{\hbar} T H_0} \psi \rangle = \langle \psi_f | e^{-\frac{i}{\hbar} T H_0} e^{\frac{i}{\hbar} T H_0} | \psi \rangle \\ &= \langle \psi_f | \psi(T) \rangle \end{aligned}$$

That's our answer. So let's do the work.

LET'S PROJECT $\psi'(T)$ INTO $\psi_f'(T)$ AND SEE WHAT WE GET

$$\langle \psi_f' | \psi'(T) \rangle = \langle \psi_f' | \psi_i' \rangle - \frac{i}{\hbar} \langle \psi_f' | \int_0^T H_i'(t) dt | \psi_i' \rangle$$

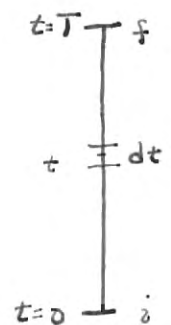
WHICH CAN BE REWRITTEN AS

$$\langle \psi_f' | \psi'(T) \rangle = \langle \psi_f' | \psi_i' \rangle - \frac{i}{\hbar} \int_0^T dt \langle \psi_f' | H_i'(t) | \psi_i' \rangle$$

THE FIRST TERM ON THE RIGHT IS JUST THE AMPLITUDE TO END UP IN THE FINAL STATE IF THERE WERE NO DISTURBANCE AROUND. USUALLY IT IS TAKEN THAT THE INITIAL AND FINAL STATES ARE ORTHOGONAL SUCH THAT $\langle \psi_f' | \psi_i' \rangle = 0$ AND WE ARE INTERESTED SOLELY IN THE SECOND TERM ON THE RIGHT. SINCE IT IS A VERY IMPORTANT CONCEPT THAT WE HAVE DEVELOPED HERE WE WILL CONTINUE TO DESCRIBE IN MORE DETAIL PHYSICALLY WHAT IS CONTAINED IN THE INTEGRAL. YOU REALLY DON'T KNOW ANYTHING WHEN YOU SEE ABSTRACT FORMULAS LIKE THE ABOVE.

TO BE FURTHER UNDERSTAND WHAT IS HAPPENING WE WILL USE SOME OF OUR PATH INTEGRAL CONCEPTS WHICH WE HAVE DEVELOPED.

LET'S DRAW A LITTLE DIAGRAM LIKE ON THE RIGHT WHERE WE START AT SOME INITIAL STATE AND ASK IT TO END UP IN SOME FINAL STATE. STARTING WITH INITIAL AMPLITUDE ψ_i THE SYSTEM DEVELOPES IN SUCH A MANNER AS TO END UP IN SOME FINAL STATE ψ_f . HOWEVER AT SOME TIME t ALONG THE WAY THERE APPEARS AN INTERACTING FORCE WHICH PRODUCES AN AMPLITUDE FOR TRANSITION FROM ONE STATE TO ANOTHER. THE TRANSITION I LIKE TO THINK OF AS A SCATTERING PROCESS. THE AMPLITUDE TO SCATTER THEN HAS A PHASE PROPORTIONAL TO $-i H_i(t) dt$ WHERE THE dt SCATTERING OCCURS OVER A RANGE OF TIME dt .



SINCE THE SCATTERING CAN TAKE PLACE AT ANYTIME BETWEEN 0 AND T, WE INTEGRATE OVER THIS RANGE. IT IS QUITE POSSIBLE THAT THE SCATTERING MAY OCCUR NOT ONCE BUT TWICE, THREE-TIME, ETC. THE INTEGRAL IS ALL INCLUSIVE OF ALL POSSIBLE INTERACTION. THE AMPLITUDE TO START IN ψ_i AND END IN ψ_f IS READ FROM RIGHT TO LEFT IN THE INTEGRAL, I.E., RIGHT TO LEFT IS THE TIME DEVELOPEMENT OF THE SYSTEM

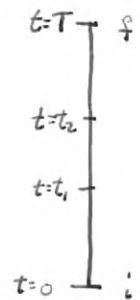
$$\langle \psi_f | \psi_i \rangle = \int_0^T \langle \psi_f | e^{-iH_0(T-t)} (-iH_i(t) dt) e^{-iH_0 t} | \psi_i \rangle$$

IF TWO SCATTERINGS OCCUR THEN WE HAVE THE SITUATION

where

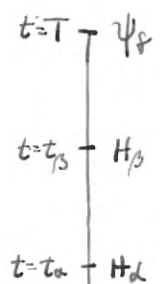
$$\langle \psi_f | \psi_i \rangle = \int_0^T \int_0^{t_2} (-i)^2 dt_2 dt_1 \langle \psi_f | e^{-i(T-t_2)H_0} H_1(t_2) e^{-i(t_2-t_1)H_0} H_1(t_1) e^{-it_1 H_0} | \psi_i \rangle$$

IN THIS CASE WE START IN STATE ψ_i AND PROPAGATE AS A FREE PARTICLE UNTIL ACTED UPON BY THE DISTURBANCE AT $t=t_1$ WHICH PRODUCES A PHASE SHIFT GIVEN BY $H_1(t_1) e^{-it_1 H_0}$. THEN THE PARTICLE CONTINUES ON UNTIL ACTED UPON BY THE SECOND SCATTERING DISTURBANCE WHICH PRODUCES ANOTHER PHASE SHIFT GIVEN BY $H_1(t_2) e^{-i(t_2-t_1)H_0}$. FINALLY THE PARTICLE PROPAGATES ON PROPORTIONAL TO $e^{-i(T-t_2)H_0}$ AS A FREE PARTICLE. THE TOTAL AMPLITUDE TO START AT i AND END AT f IS GIVEN BY THE PRODUCT OF THE INDIVIDUAL AMPLITUDES. THE DIAGRAM, COMMONLY CALLED A FEYNMAN DIAGRAM AFTER THE GUY WHO FIRST DEVELOPED THEIR USE, IS JUST A SIMPLE WAY TO SAVE YOU ALL THE TIME GRINDING THROUGH THESE MESSY INTEGRALS. THERE ISN'T ANYTHING INTUITIVELY MYSTERIOUS ABOUT THEIR FORM ITS JUST ANOTHER USEFUL TOOL IN DESCRIBING THESE KINDS OF PROCESSES.



YOU HAVE TO BE CAREFUL THOUGH IN APPLYING THE INTEGRAL EXPRESSION BECAUSE TIME ORDER IS CRITICAL IN DESCRIBING WHICH SCATTERING OCCURS WHEN. AS EXAMPLE CONSIDER A PROCESS DESCRIBED BY THE DIAGRAM ON THE RIGHT WHERE THERE ARE NOW TWO DIFFERENT TYPES OF DISTURBANCES OCCURRING, TYPES H_α AND H_β , SAY. H_β MUST OCCUR AFTER H_α CHRONOLOGICALLY AS DEPICTED. THEREFORE, THE INTEGRAL MUST REFLECT THIS TIME ORDERING, I.E.,

$$\langle \psi_f | \psi_i \rangle = (-i)^2 \int_0^T \int_0^{t_\beta} dt_\alpha dt_\beta \langle \psi_f | e^{-iH_0(T-t_\beta)} H_\beta(t_\beta) e^{-iH_0(t_\beta-t_\alpha)} H_\alpha(t_\alpha) e^{-iH_0 t_\alpha} | \psi_i \rangle$$



THIS IS THE AMPLITUDE IF H_α OCCURS BEFORE H_β BUT THERE IS ALSO THE POSSIBILITY THAT H_β OCCURS BEFORE H_α . THAT CASE IS GIVEN BY

$$(-i)^2 \int_0^T \int_{t_\beta}^t dt_\alpha dt_\beta \langle \psi_f | e^{-iH_0(T-t_\alpha)} H_\alpha(t_\alpha) e^{-iH_0(t_\alpha-t_\beta)} H_\beta(t_\beta) e^{-it_\beta H_0} | \psi_i \rangle$$

THE TOTAL AMPLITUDE IS THE SUM OF THESE TWO AMPLITUDES IF IT IS NOT KNOWN WHETHER H_α WILL OCCUR BEFORE H_β .

38. INTERACTION OF AN ATOM WITH AN ELECTROMAGNETIC FIELD

LAST TIME WE LEARNED HOW THE HAMILTONIAN OF A SYSTEM COULD BE BROKEN INTO TWO PARTS - ONE EASY PIECE, H_0 , AND ANOTHER PIECE, H_1 , WHICH IS A MORE COMPLICATED FUNCTION OF TIME DESCRIBING THE INTERACTION BETWEEN THE PARTS OF THE SYSTEM. THE RULE WHICH WE DEVELOPED WAS THAT THE AMPLITUDE TO SCATTER IN TIME dt IS PROPORTIONAL TO $-iH_1(t) dt$. THIS REPRESENTS AN EXTRA PIECE OF THE ANSWER WHICH IS A CORRECTION TO THE H_0 PIECE WHICH IS FOR NON-INTERACTING PARTS. THE TRANSITION OF A SYSTEM FROM SOME INITIAL STATE i TO SOME FINAL STATE f WAS DESCRIBABLE BY THE APPROXIMATIONS:

$$\begin{aligned} 1^{\text{st}} \text{ order} & \quad \langle \psi_f | e^{-iH_0 T} | \psi_i \rangle \\ 2^{\text{nd}} \text{ order} & \quad \langle \psi_f | e^{-iH_0(T-t)} \left(-\frac{i}{\hbar} H_1(t) dt \right) e^{-iH_0 t} | \psi_i \rangle \end{aligned}$$

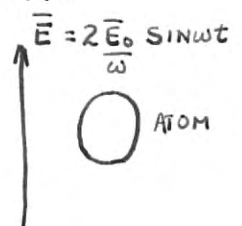
THE 1st ORDER TERM IS THE AMPLITUDE TO GO FROM i TO f WITHOUT ANY INTERACTION IN TIME T . THE SECOND ORDER TERM CORRECTS THE RESULT FOR A SCATTERING WHICH OCCURS AT TIME t IN TRANSIT FROM i TO f . THE COMPLETE ANSWER MUST INCLUDE THE POSSIBILITY THAT THE SCATTERING CAN OCCUR ANYWHERE BETWEEN i AND f OR BETWEEN $0 \leq t \leq T$. THE ANSWER IS THEN

$$\langle \psi_f | \psi_i \rangle = \langle \psi_f | e^{-iH_0 T} | \psi_i \rangle + \int_0^T \langle \psi_f | e^{-iH_0(T-t)} \left(-\frac{i}{\hbar} H_1(t) dt \right) e^{-iH_0 t} | \psi_i \rangle$$

WE WOULD LIKE TO APPLY THIS RESULT TO SOLVE SOME PROBLEMS WHICH ONLY REQUIRE THE FIRST ORDER CORRECTION TERM. LATTER WE'LL GET MORE COMPLICATED AND CORRECT TO SECOND ORDER SCATTERING.

THE PROBLEM WE WILL CONSIDER IS AN ATOM IN AN ELECTRIC FIELD \vec{E} . \vec{E} IS SINUSOIDALLY VARYING AT FREQUENCY ω AND IS THEREFORE DESCRIBABLE BY A VECTOR POTENTIAL \vec{A} , WHERE

$$\frac{\partial \vec{A}}{\partial t} = \vec{E} \rightarrow \vec{A} = 2A_0 \cos \omega t, \quad A_0 = \frac{E_0}{\omega}$$



WE WILL FIRST CONSIDER A FREQUENCY SUFFICIENTLY LOW THAT THE WAVELENGTH IS LONG RELATIVE TO THE ATOMIC DIMENSION; THUS WE DON'T HAVE TO WORRY ABOUT VARIATIONS OF \vec{A} ACROSS THE ATOM.

TO BEGIN THE PROBLEM IT IS ASSUMED THAT SOMEHOW WE KNOW THE ENERGY STATES OF THE ATOM IN THE ABSENCE OF THE EXTERNAL FIELD. IN PRINCIPAL WE CAN IN FACT CALCULATE THE ENERGY STATES OF AN UNPERTURBED ATOM BUT IT IS QUITE DIFFICULT AS WE HAVE POINTED OUT EARLIER. CONSIDER CHROMIUM, FOR INSTANCE, WITH 24 ELECTRONS. THIS IS A VERY COMPLICATED STRUCTURE WHICH REQUIRES MANY INDEPENDENT VARIABLES: 3 FOR EACH ELECTRON AND 3 FOR THE NUCLEUS OR 75 TOTAL. BUT NEVERTHELESS WE ASSUME H_0 IS SOLVED AND ALL THE EIGENFUNCTIONS ψ_n HAVE BEEN WORKED OUT SO THAT

$$H_0 \psi_n = E_n \psi_n$$

IS KNOWN. THE ψ_n 'S THEN DESCRIBE THE BEHAVIOR OF ALL THE ELECTRONS WITHIN THE ATOM ITSELF. NOW WE WANT TO KNOW WHAT HAPPENS WHEN THE \vec{E} -FIELD IS PRESENT AND HOW H_0 CHANGES. FOR THE LEVEL OF ACCURACY HERE WE WILL CONSIDER THE CHANGE TO H_0 IS PROPORTIONAL TO \vec{E} . THERE IS NO POINT IN GOING TO SECOND ORDER E TERMS SINCE WE ARE ONLY CONCERNED WITH FIRST ORDER VARIATIONS IN H_0 . THIS IS A KEY POINT IN SIMPLIFYING THE WORK TREMENDOUSLY SO I SUGGEST YOU GET USE TO MAKING THESE APPROXIMATIONS YOURSELVES.

WE NOW MUST WRITE THE HAMILTONIAN FOR THE SYSTEM. WE KNOW THAT $H = \frac{p^2}{2m}$ IS NOT THE COMPLETE HAMILTONIAN WHEN AN ELECTRIC FIELD IS PRESENT. THE CORRECT HAMILTONIAN FOR A PARTICLE IN A VECTOR POTENTIAL \vec{A} IS

$$H = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} = \frac{1}{2m} \left(-\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A} \right) \cdot \left(-\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A} \right)$$

THIS HAMILTONIAN THEN WORKS ON ψ TO PRODUCE THE ENERGY STATES OF THE SYSTEM. EXPANDING OUT $H\psi$ WE HAVE THAT

$$H\psi = \frac{1}{2m} (-\hbar^2 \nabla^2) \psi + \frac{1}{2m} \left[\frac{\hbar e}{i c} \vec{\nabla} \cdot (\vec{A}\psi) \right] + \frac{1}{2m} \left[\frac{\hbar e}{i c} \vec{A} \cdot (\vec{\nabla}\psi) \right] + \frac{1}{2m} \left(\frac{e^2}{c^2} \right) (\vec{A} \cdot \vec{A}) \psi$$

I should point out here, before I continue, if the particle (atom, molecule, etc) is made up of a lot of electrons, we must keep track of them all in the Hamiltonian, i.e.,

$$H_0 = \sum_j \frac{p_j^2}{2m_j} + \sum_j \sum_k V(r_{jk})$$

where $V(r_{jk})$ is the interaction potential between the j and k particles. The total Hamiltonian is

$$H = \sum_j \frac{(\vec{p}_j - \frac{e}{c} \vec{A}(x_j, y_j, z_j))^2}{2m_j} + \sum_j \sum_k V(r_{jk})$$

The vector potential must be evaluated at each of the j^{th} particle.

Now we notice in the expanded form of H three terms coming in which involve \vec{A} . These are the extra pieces which change our results from the case of no external field, i.e., the H_0 piece which is $-\frac{\hbar^2}{2m} \nabla^2 \psi$. Since $H_1(t)$ lacks physical interpretation we will simply compute it by taking the difference between H and H_0 . In other words

$$H_1(t) = H - H_0$$

or

$$H_1(t) = \sum_j \frac{e_j}{2m_j c} [\vec{p}_j \cdot \vec{A}(x_j) + \vec{A}(x_j) \cdot \vec{p}_j] + \sum_j \frac{e_j^2}{2m_j} \vec{A}(x_j) \cdot \vec{A}(x_j)$$

If we take the freedom to select the gauge transformation $\vec{\nabla} \cdot \vec{A} = 0$ (i.e., the plane wave is assumed transverse) then

$$\vec{p} \cdot \vec{A} = \vec{A} \cdot \vec{p}$$

Proof:

$$(\vec{\nabla} \cdot (\vec{A} \psi)) = (\vec{A} \cdot \vec{\nabla}) \psi$$

$$(\vec{\nabla} \cdot \vec{A}) \psi + \vec{A} \cdot \vec{\nabla} \psi = \vec{A} \cdot \vec{\nabla} \psi$$

If we choose a different gauge we must include a scalar potential χ in the Hamiltonian.

One other point we agree that A_0 is a small potential and that terms in A_0^2 can be ignored since we are not computing terms in H_1^2 . Therefore we have that

$$H_1(t) = \sum_j \frac{e_j}{m_j c} \vec{p} \cdot \vec{A}$$

SINCE WE HAVE CHOSEN A GAGE WHICH MAKES \vec{A} TRANSVERSE WE WILL DEFINE THE DIRECTION OF A_0 TO BE THE X DIRECTION SUCH THAT

$$\vec{A} = A \vec{e}_x = 2A_0 x \cos \omega t$$

THEN THE OPERATOR $\vec{p} = \frac{\hbar}{i} \frac{\partial}{\partial x_j}$

I WOULD LIKE TO DEFINE A NEW OPERATOR $\hat{j}_x = \frac{e_j P x_j}{m_j c}$ WHICH IS NOT TO BE CONSIDERED A CURRENT IN THE CLASSICAL SENSE BUT RATHER A TRUE OPERATOR WORKING ON ψ .

NOW WE ARE READY TO GO TO WORK. FIRST WE WILL TAKE THE ZERO ORDER TERM WHICH DESCRIBES THE BEHAVIOR OF THE SYSTEM WITH NO EXTERNAL FIELD. OF COURSE, WE KNOW THE ANSWER ALREADY,

$$\langle \psi_f | e^{-i \hat{H}_0(t)} | \psi_i \rangle$$

CONSIDER THE SYSTEM INITIALLY IN STATE n AND SUPPOSE WE WANT TO KNOW WITH WHAT AMPLITUDE IT WILL BE FOUND IN STATE m AT TIME T . THEN

$$\langle \phi_m | \phi_n \rangle = \langle \phi_m | e^{-i H_0(t)} | \phi_n \rangle$$

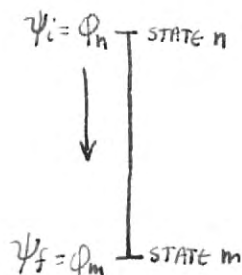
BUT WE KNOW WHAT $e^{-i H_0 T}$ ON $|\phi_n\rangle$ IS IT IS JUST $e^{-i E_n T} |\phi_n\rangle$ AND THEREFORE OUR ANSWER IS

$$e^{-i E_n T} \langle \phi_m | \phi_n \rangle$$

GENERALLY WE WANT THE TRANSITION TO OCCUR BETWEEN TWO EIGENSTATES WHICH IMPLIES BY ORTHOGONALITY THAT

$$\langle \phi_m | \phi_n \rangle = 0$$

IF THE STATES ARE NOT DEGENERATE. THE WHOLE PHYSICAL PROBLEM THUS REDUCES TO EVALUATING THE 1^{ST} ORDER TERM WHICH WE WILL NOW DO.



TRANSITION AMPLITUDE

The amplitude to go from state n to state m in the presence of a scattering potential is

$$S_{mn} = \int_0^T \langle \varphi_m | e^{-iH_0(T-t)} \left[-\frac{i}{\hbar} H_1(t) e^{-iH_0 t} \right] | \varphi_n \rangle dt$$

We can start to evaluate the integral by letting $e^{-iH_0 t}$ work on the states $|\varphi_m\rangle$ and $|\varphi_n\rangle$.

$$e^{-iH_0 t} |\varphi_n\rangle = e^{-iE_n t} |\varphi_n\rangle$$

$$\langle \varphi_m | e^{iH_0(T-t)} = \langle \varphi_m | e^{iE_m(T-t)}$$

The last expression coming from the Hermitian property of φ_m . Also $e^{-iE_m T}$ can be brought outside the integral since it is constant with time. Therefore we have simplified the integral to,

$$S_{mn} = -\frac{i}{\hbar} e^{-iE_m T} \int_0^T dt e^{i\frac{E_m - E_n}{\hbar} t} \langle \varphi_m | H_1 | \varphi_n \rangle$$

The probability to make a transition between n and m is thus proportional to the square of the matrix element $\langle \varphi_m | H_1 | \varphi_n \rangle$ which must be integrated. The constant phase factor $e^{-i\frac{E_m}{\hbar} T}$ goes out in the squaring.

$$P(n \rightarrow m) \propto |S_{mn}|^2 = \left| \int_0^T dt e^{i\frac{E_m - E_n}{\hbar} t} \langle \varphi_m | H_1 | \varphi_n \rangle \right|^2$$

What this result says physically is that probability to make a transition between two states depends on the amount of $H_1(t)$ Fourier component contained in the frequency range as defined by the difference between the energy levels,

$$\omega_{mn} = \frac{E_m - E_n}{\hbar}$$

If $H_1(t)$ has a ~~strong~~ large frequency component corresponding to transition frequency then the perturbation or scatter is likewise strong. Another way to think of this phenomena is to draw an analog with the power density of H_1 in the appropriate frequency range. That is the whole story of transition amplitudes and it is quite pretty and simple; it should be learned by the students earlier than it is taught since all of classical physics stems from the concept.

WE SHOULD NOW SPECIALIZE THE ANALYSIS TO THE PROBLEM WE WOULD WISH TO SOLVE BY PUTTING IN OUR EXPRESSION FOR H_1

$$H_1 = 2j_x A_0 \cos \omega t$$

IF WE PUT THIS INTO S_{mn} WE CAN BRING OUT OF THE INTEGRAL THE TIME INDEPENDENT MATRIX ELEMENT $\langle \phi_m | j_x | \phi_n \rangle$ AND GET

$$S_{mn} = -\frac{i}{\hbar} e^{-i \frac{E_m T}{\hbar}} \langle \phi_m | j_x | \phi_n \rangle \int_0^T dt e^{i \frac{(E_m - E_n)t}{\hbar}} 2 A_0 \cos \omega t$$

IN ORDER TO GET AN ANSWER WE WILL SUBSTITUTE FOR $\cos \omega t$

$$\cos \omega t = \frac{1}{2} (e^{i \omega t} + e^{-i \omega t})$$

AND STREAMLINE THE NOTATION BY LETTING $(j_x)_{mn} = \langle \phi_m | j_x | \phi_n \rangle$.

THE EXPRESSION FOR S_{mn} THEN BECOMES INTEGRABLE

$$S_{mn} = -\frac{i}{\hbar} e^{-i \frac{E_m T}{\hbar}} (j_x)_{mn} \int_0^T A_0 \left[e^{i \left(\frac{E_m - E_n}{\hbar} + \omega \right) t} + e^{i \left(\frac{E_m - E_n}{\hbar} - \omega \right) t} \right] dt$$

IF WE INTEGRATE AND CONSIDER A_0 CONSTANT

$$S_{mn} = -\frac{i}{\hbar} e^{-i \frac{E_m T}{\hbar}} (j_x)_{mn} \left[\frac{A_0 e^{i \left(\frac{E_m - E_n}{\hbar} + \omega \right) T}}{\left(\frac{E_m - E_n}{\hbar} + \omega \right)} + \frac{A_0 e^{i \left(\frac{E_m - E_n}{\hbar} - \omega \right) T}}{\left(\frac{E_m - E_n}{\hbar} - \omega \right)} \right]$$

THE MATRIX ELEMENT $(j_x)_{mn}$ IS A COMPLEX NUMBER WHICH HAS TO BE WORKED OUT FOR EACH PROBLEM TO BE SOLVED. FOR NOW WE WILL LEAVE IT JUST AS A MATRIX ELEMENT AND ASSUME IT CAN BE CALCULATED.

BY EXAMINING S_{mn} YOU SEE THAT IF A_0 IS WEAK YOU WON'T GET MUCH AMPLITUDE TO SCATTER. UNLESS, HOWEVER, THE DENOMINATORS GET QUITE SMALL. FOR SUCH A CASE S_{mn} CAN GET BIG EVEN IF A_0 IS WEAK BY HAVING THE DENOMINATOR GET CLOSE TO ZERO. IF THE EXTERNAL FIELD IS FLUCTUATING AT FREQUENCY ω AND IT TURNS OUT THAT

$$\omega = \pm \left(\frac{E_m - E_n}{\hbar} \right) = \pm \omega_{mn}$$

THEN THE ATOM IS EXCITED AT ITS RESONANT TRANSITION FREQUENCY BETWEEN STATE n AND m , AND THE SYSTEM REALLY RUNS UP OR DOWN DEPENDING UPON THE SIGN OF THE ENERGY DIFFERENCE. IF $E_m > E_n$ AND WE ASK WHAT IS THE TRANSITION AMPLITUDE TO GO FROM n TO m , WE ARE CONSIDERING RESONANT ABSORPTION. WHILE FOR $E_m < E_n$ WE DRIVE THE SYSTEM DOWN AND HAVE RESONANT EMISSION.

NOTICE ALSO FROM THE FORM OF THE TERMS IN THE BRACKET EITHER ONE OR THE OTHER TERM IS IMPORTANT. IF $\omega = -\omega_{mn}$ THE FIRST TERM IS IMPORTANT WHILE THE SECOND TERM JUST SITS THERE RAPIDLY OSCILLATING NEVER CONTRIBUTING MUCH TO THE EMISSION PROCESS. FOR THAT REASON WE WILL FORGET THE SECOND TERM BY LIMITING OURSELVES TO THE CASE FOR EMISSION. IT SIMPLIFIES THE CALCULATION SO YOU DON'T GET LOST.

THE ANSWER IS NOW AT HAND FOR WE WANT THE SQUARE OF S_{mn} TO GET THE PROBABILITY THAT THE SYSTEM WILL MAKE THE TRANSITION OF INTEREST.

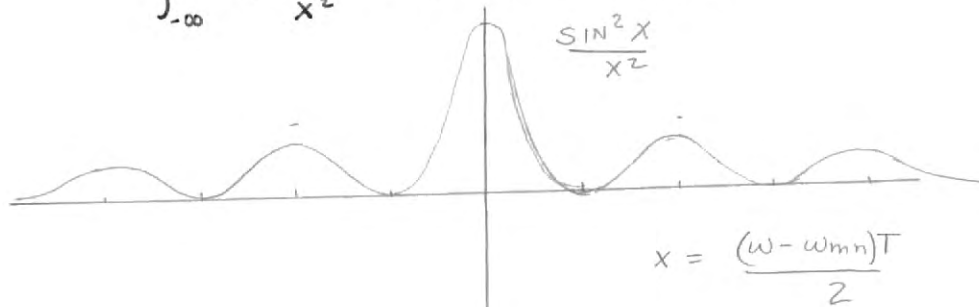
$$P(n \rightarrow m) = |S_{mn}|^2 = \frac{1}{\hbar^2} |(jx)_{mn}|^2 |A_0|^2 \left[\frac{e^{i(\omega - \omega_{mn})T} - 1}{\omega - \omega_{mn}} \right]^2$$

THE SQUARED BRACKET CAN BE EXPANDED OUT AND SIMPLIFIED TO GIVE THE ANSWER

$$P(n \rightarrow m) = \frac{1}{\hbar^2} |(jx)_{mn}|^2 |A_0|^2 \frac{4 \sin^2(\omega - \omega_{mn}) \frac{T}{2}}{(\omega - \omega_{mn})^2}$$

THE EXPRESSION $\frac{4 \sin^2(\omega - \omega_{mn}) \frac{T}{2}}{(\omega - \omega_{mn})^2}$ IS VERY LARGE WHEN RESONANCE IS APPROACHED, I.E., NEAR $\omega = \omega_{mn}$ THE PROBABILITY IS PROPORTIONAL TO T^2 . THE SHARP RESONANCE WHICH DEMONSTRATES THIS T^2 DEPENDENCE IS IN PRACTICE IS VERY DIFFICULT TO DUPLICATE IN THE LAB. PROBLEMS INVOLVING TURNING ON A_0 AND TURNING IT OFF INTRODUCE OTHER FREQUENCY COMPONENTS WHICH DEGENERATE THE SPIKE OF THE RESONANCE. THE PROBABILITY DISTRIBUTION DEPENDS ON THE FUNCTION $\frac{\sin^2 x}{x^2}$ A LITTLE LESS STRONGLY THAN INDICATED WHEN AN AVERAGE OVER THE A LIMITED FREQUENCY RANGE IS PERFORMED. THE ACTUAL AVERAGED DEPENDENCE IS PROPORTIONAL TO $\delta(\omega - \omega_{mn}) T$

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = 2\pi \delta(\omega - \omega_{mn}) T$$



The bumpy character of the transition probability is quite similar to the diffraction pattern from a slit. The bumpy character can be observed if all the aspects of the experiment are tightly controlled. Practically the bumpiness is lost and the curve dies off as the average of the sinusoidally damped curve.

Since A_0 in practice has ~~em~~ frequency components other than the resonant frequency it is necessary to integrate over the expected frequency range or bandwidth. If we think in terms of the mean value of A_0 to be analogous to the power contained in the field then

$$\overline{A_0^2} = \text{Power } (\omega)$$

And,
$$P(n \rightarrow m) = \int \frac{2\pi}{\hbar} |(\hat{j}_x)_{mn}|^2 P(\omega) \delta(\omega - \omega_{nm}) T d\omega$$

which is quickly integrated to get

$$P(n \rightarrow m) = \frac{2\pi}{\hbar} |(\hat{j}_x)_{mn}|^2 P(\omega_{mn}) T$$

The probability per unit time becomes proportional to the intensity or power of the field which is concentrated at frequency ω_{mn} . The proportionality constant depends on the complex matrix elements which in turn depends on the internal structure of the atom. $|(\hat{j}_x)_{mn}|$ has the property of a hermitian operator such that

$$(\hat{j}_x)_{mn} = (\hat{j}_x)_{nm}$$

which tells us that the odds of driving the system up are just as good as driving us down.

If the transition occurs into a continuum of energy state or at least extremely closely spaced states, then we must sum over the density of states $\rho(E)$ or else integrate.

$$\sum_{m=1}^{\infty} P(n \rightarrow m) = \sum \frac{2\pi}{\hbar} |(\hat{j}_x)_{mn}|^2 \delta(\omega - \omega_{mn}) |A_0|^2$$

If we have the ATOM BEING ILLUMINATED WITH A LIGHT WHICH IS NOT MONOCHROMATIC THEN THE POWER OR INTENSITY $I(\omega)$ AT FREQUENCY ω_{mn} PRODUCES THE PROBABILITY PER SECONDS OF

$$\begin{aligned} \frac{P(n \rightarrow m)}{\text{SEC}} &= \frac{2\pi}{\hbar} \left| \langle \psi_x \rangle_{mn} \right|^2 \frac{I(\omega_{mn})}{(\omega_{mn})^2} \\ &= \frac{2\pi}{\hbar} \left| \frac{\langle \psi_x \rangle_{mn}}{\omega_{mn}} \right|^2 I(\omega_{mn}) \end{aligned}$$

THE STRENGTH OF THE TRANSITION, ~~IS~~ GIVEN BY THE SQUARE MATRIX ELEMENT, IS SOMETIMES CALLED THE DIPOLE MATRIX ELEMENT. SINCE THE TRANSITION PROBABILITY IS PROPORTIONAL TO THE TIME THE RESULT BECOMES LARGE ONLY FOR A NARROW BANDWIDTH OF FREQUENCIES CENTERED AT ω_{mn} . THE BANDWIDTH $\Delta\omega$ IS ON THE ORDER OF $1/T$. THUS THE LONGER YOU WAIT THE CLOSER YOU MUST EXCITE THE ATOM AT RESONANCE. ONE ADDITIONAL COMMENT THE TOTAL PROPORTIONALITY CONSTANT $\frac{2\pi}{\hbar} \left| \frac{\langle \psi_x \rangle_{mn}}{\omega_{mn}} \right|^2$ IS SOMETIME CALLED THE EINSTEIN TRANSITION ELEMENT.

39. COULOMB EXCITATION

LAST TIME WE ENDED UP TALKING ABOUT MAKING TRANSITIONS IN A CONTINUUM OF ENERGY STATES. WE SAW THAT IT WAS NECESSARY TO SUM AVER ALL POSSIBLE STATES IN THE CONTINUUM SINCE IT IS MEANINGLESS TO TALK ABOUT A TRANSITION TO ONE STATE IN THE CONTINUUM. THE RESULT WE OBTAINED THEN WAS

$$\sum_m \frac{P(n \rightarrow m)}{\text{SEC}} = \sum_m \frac{2\pi}{\hbar} |A_{mn}|^2 \delta(E_m - E_n - \hbar\omega)$$

WHERE A_{mn} IS THE MATRIX ELEMENT OF THE PERTURBATION, $H_1 = 2A \cos \omega t$; IT IS THE VECTOR POTENTIAL. IF WE CAN ASSUME THE CONTINUUM IS DESCRIBABLE BY A DENSITY FUNCTION $\rho(E_m)$ THEN WE CAN INTEGRATE OVER ALL STATES E_m

$$\begin{aligned} \sum_m \frac{P(n \rightarrow m)}{\text{SEC}} &\rightarrow \int_{E_n}^{E_n + \hbar\omega} \frac{2\pi}{\hbar} |A_{mn}|^2 \delta(E_m - E_n - \hbar\omega) \rho(E_m) dE_m \\ &= \frac{2\pi}{\hbar} |A_{mn}|^2 \rho(E) \end{aligned}$$

WHERE $E = E_n + \hbar\omega$. THIS THEN IS THE PROBABILITY PER UNIT TIME TO MAKE A TRANSITION TO SOME ARBITRARY STATE IN THE CONTINUUM.

ATOMIC IONIZATION

I'D LIKE TO WORK OUT A PROBLEM IN WHICH THE PERTURBING POTENTIAL H_1 IS NOT A FUNCTION OF TIME. IN THIS SPECIAL CASE THE PROBABILITY TO MAKE A TRANSITION PER SECOND IS JUST

$$\frac{P(n \rightarrow m)}{\text{SEC}} = \frac{2\pi}{\hbar} |(H_1)_{mn}|^2 \delta(E_m - E_n)$$

THE PROBLEM INVOLVES A HIGHLY EXCITED ATOM A AND ANOTHER ATOM B WHICH IS NEARBY AND WE WANT TO KNOW WITH WHAT PROBABILITY THE EXCITED ATOM INDUCES IONIZATION IN THE OTHER. THE HAMILTONIAN OF THE SYSTEM IS JUST

$$H = H_0 + \frac{e^2}{R_{ab}}$$

WHERE $H_0 = H_A + H_B$ THE SUM OF THE HAMILTONIANS OF THE NON-INTERACTING ATOM. THE COULOMB POTENTIAL IS A FUNCTION OF THE ELECTRON SEPARATION DISTANCE ONLY AND DOES NOT DEPEND ON TIME.

INITIALLY ATOM B IS IN THE GROUND STATE DESCRIBABLE BY THE WAVE FUNCTION ϕ_B . ATOM A IS IN THE EXCITED STATE p , ϕ_p^A . THEREFORE THE INITIAL WAVE FUNCTION OF THE SYSTEM ϕ_{in} IS

$$\phi_{in} = \phi_0^B \phi_p^A$$

THE FINAL STATE FINDS ATOM B IONIZED WITH ITS ELECTRON MOVING WITH MOMENTUM k WHILE ATOM A HAS FALLEN TO ITS GROUND STATE q . THEREFORE THE FINAL STATE ϕ_{fin} OF THE SYSTEM IS

$$\phi_{fin} = \phi_k^B \phi_q^A$$

THE RESPECTIVE ENERGY STATES INITIALLY AND FINALLY ARE

$$E_m = E_0^B + E_p^A$$

AND

$$E_n = E_k^B + E_q^A$$

ALL WE HAVE TO DO TO FINISH THE PROBLEM IS TO COMPUTE THE INTEGRAL

$$\int \langle \phi_{fin} | (H_1)_{mn} | \phi_{in} \rangle dV_{ol} = \int \phi_q^A \phi_k^B \frac{e^2}{r_{ab}} \phi_0^B \phi_p^A d^3r_a d^3r_b$$

THE PROBABILITY TO MAKE THE TRANSITION IS JUST

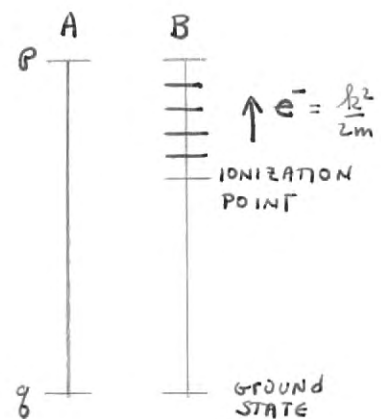
$$\frac{P(m \rightarrow n)}{\delta \epsilon} = \int_0^{E_k} 2\pi |H_{11}|^2 \delta(E_k + E_q^A - E_0^B - E_p^B) \rho(E_k) dE_k$$

WHERE WE INTEGRATE OVER ALL POSSIBLE MOMENTUM VALUES.

TO WORK OUT THE INTEGRAL WE KNOW THAT THE ELECTRON IN A CONTINUUM HAS A WAVE FUNCTION PROPORTIONAL TO e^{ikx} THUS $\phi_k^B \propto e^{ikx}$, I.E. THE ELECTRON IS FREE. THE INTEGRATION IS TAKEN OVER ALL POSSIBLE MOMENTUMS SO THAT

$$\rho(E_k) dE_k \rightarrow \frac{d^3k}{(2\pi)^3}$$

AND THE INTEGRATION IS NOW IN k -SPACE WHICH IS EASIER TO WORK WITH. FOR THE MOMENT WE ARE GOING TO AVERAGE H_1 OVER ALL DIRECTIONS OF k AND TAKE IT OUT OF THE INTEGRAL.



WE NOW HAVE,

$$\frac{P(m \rightarrow n)}{\text{SEC}} = \frac{2\pi}{\hbar} \overline{|(H_i)_{mn}|^2} \int \delta\left(\frac{\hbar^2 k^2}{2m} + E_0^A - E_0^B - E_P^A\right) \frac{d^3 k}{(2\pi)^3}$$

LETS DEFINE $E_0^A - E_0^B - E_P^A$ AS $-W$ AS SINCE IT IS THE EXCESS ENERGY REQUIRED OVER THE IONIZATION LEVEL; E_0^B IS DEFINED HERE FROM THE IONIZATION POINT SO IT IS A NEGATIVE ENERGY. IF WE CONSIDER THE k -SPACE TO BE SPHERICALLY SYMMETRIC THEN

$$\frac{P(m \rightarrow n)}{\text{SEC}} = \frac{2\pi}{\hbar} \overline{|H_i|^2} \int \frac{4\pi k^2}{(2\pi)^3} \delta\left(\frac{\hbar^2 k^2}{2m} - W\right) dk$$

TO EXPLICITLY EVALUATE THE INTEGRAL WE HAVE TO USE THE PROPERTY OF A DELTA FUNCTION THAT

$$\delta(f(x)) dx = \frac{1}{|f'(x)|}$$

PERFORMING THE DIFFERENTIATION AND SIMPLIFYING WE HAVE

$$\frac{P(m \rightarrow n)}{\text{SEC}} = \frac{2}{\hbar \pi} \overline{|H_i|^2} m k$$

WHERE $k = \sqrt{2mW}$. THAT'S THE FINAL RESULT EXCEPT FOR THE FACT THAT WE NEVER EVALUATED H_i . USUALLY IT IS TOO DIFFICULT TO MAKE AN EXPLICIT EVALUATION BUT SOMETIME IT IS POSSIBLE TO MAKE AN APPROXIMATION.

IF THE ATOMS ARE FAR APART WE CAN APPROXIMATE THE ELECTRON SEPARATION DISTANCE R_{ab} BY

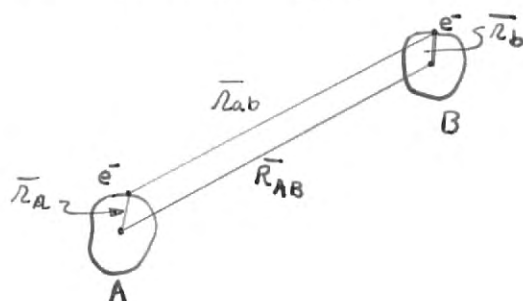
$$R_{ab} = R_{AB} - \vec{r}_a \cdot \vec{R}_{AB} + \vec{r}_b \cdot \vec{R}_{AB}$$

WHERE R_{AB} IS THE DISTANCE BETWEEN THE TWO NUCLEI AND \vec{r}_a AND \vec{r}_b ARE THE ELECTRON'S LOCATION RELATIVE TO THEIR RESPECTIVE NUCLEUS. THUS WE HAVE

$$\frac{1}{R_{ab}} = \frac{1}{R_{AB}} + \frac{1}{R_{AB}^3} (\vec{r}_a \cdot \vec{R}_{AB} - \vec{r}_b \cdot \vec{R}_{AB})$$

THEREFORE

$$\langle \phi_n | H_i | \phi_m \rangle = \int \phi_0^A \phi_k^B e^2 \left[\frac{1}{R_{AB}} + \frac{1}{R_{AB}^3} (\vec{r}_a \cdot \vec{R}_{AB} - \vec{r}_b \cdot \vec{R}_{AB}) \right] \phi_0^B \phi_P^A d^3 r_a d^3 r_b$$



IF WE INTEGRATE OUT THE FIRST TERM, WE FIND THAT IT IS ZERO. THIS IS A CONSEQUENCE OF THE INTEGRATION OVER $d^3 r_a$, I.E.,

$$\frac{1}{R_{AB}} \int \varphi_g^A(a) \varphi_p^A(a) d^3 r_a = 0$$

BECAUSE φ_g^A AND φ_p^A ARE ORTHOGONAL EIGENFUNCTIONS AS WE CONSTRUCTED THE PROBLEM. THE SECOND TERM

$$\int \int \varphi_g^A(a) \varphi_k^B(b) \left(\frac{\bar{r}_a \cdot \bar{r}_{AB}}{R_{AB}^3} \right) \varphi_0^B(b) \varphi_p^A(a) d^3 r_b d^3 r_a$$

AGAIN PRODUCES 0 WHEN INTEGRATING OVER $d^3 r_b$ BECAUSE THE FUNCTION $\bar{r}_a \cdot \bar{r}_{AB} / R_{AB}^3$ DOES NOT DEPEND ON r_b . BY A SIMILAR APPROACH THE THIRD TERM $\propto (-\bar{r}_b \cdot \bar{r}_{AB} / R_{AB}^3)$ WHEN INTEGRATED OVER $d^3 r_a$ GIVES ZERO. SO THE ANSWER IS NOTHING! NO NOT QUITE. OBVIOUSLY OUR APPROXIMATION IS NOT GOOD ENOUGH SO WE HAVE TO GO TO SECOND ORDER EXPANSIONS OF $1/r_{ab}$. EXPANDING TO THIRD ORDER WE HAVE

$$\frac{1}{r_{ab}} = \frac{1}{R_{AB}} + \frac{1}{R_{AB}^2} \left(\frac{\bar{r}_{ab} \cdot \bar{r}_{AB}}{R_{AB}} - \frac{\bar{r}_b \cdot \bar{r}_{AB}}{R_{AB}} \right) + \frac{x_a x_b + y_a y_b - 2 z_a z_b}{R_{AB}^3}$$

FOR THE FIRST NEW TERM

$$\int \int_{r_b, r_a} \varphi_g^A(a) \varphi_k^B(b) e^2 \frac{x_a x_b}{R_{AB}^3} \varphi_0^B(b) \varphi_p^A(a) d^3 r_a d^3 r_b$$

TO EVALUATE THIS INTEGRAL WE'LL SEPARATE IT IN THE FOLLOWING WAY

$$(H_{ij})_{mn} = \frac{1}{R_{AB}^3} \left[\int \varphi_g^A(a) e x_a \varphi_p^A(a) d^3 r_a \right] \left[\int \varphi_k^B(b) e x_b \varphi_0^B(b) d^3 r_b \right]$$

THE BRACKETED QUANTITIES ARE KNOWN AS THE ELECTRIC DIPOLE MATRIX ELEMENTS

$$\int \varphi_g^A(a) e x_a \varphi_p^A(a) d^3 r_a = (e x_{pg})^A$$

$$\int \varphi_k^B(b) e x_b \varphi_0^B(b) d^3 r_b = (e x_{0k})^B$$

THE OTHER DIRECTIONS Y AND Z PRODUCE SIMILAR DIPOLE TERMS WHICH CAN BE AVERAGED OVER ALL DIRECTIONS TO PRODUCE AN AVERAGE DIPOLE MATRIX ELEMENT. SO OUR FINAL RESULT IS

$$\frac{P(m \rightarrow n)}{\text{SEC}} \propto \frac{1}{R_{AB}^6} |x^A|^2 |x^B|^2 m k$$

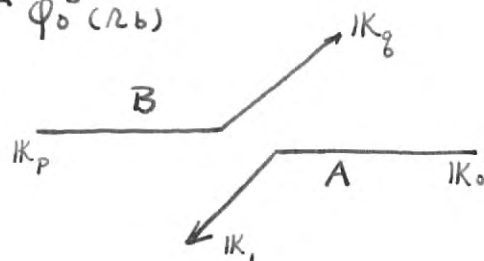
If we continued this problem a little further by complicating the interaction we can consider what happens if the two atoms are in motion relative to each other. We want the probability to make a transition as the excited atom flies by. The Hamiltonian of the system as viewed from the center of mass is

$$H = \underbrace{H_A + H_B + \frac{P_A^2}{2m_A} + \frac{P_B^2}{2m_B}}_{H_0} + \underbrace{\frac{e^2}{|\vec{r}_A + \vec{R}_A - (\vec{r}_B + \vec{R}_B)|}}_{H_1}$$

The initial state of the system has a wave function and energy of

$$\psi_{in} = e^{i\vec{k}_p \cdot \vec{R}_B} \phi_p^A(r_a) e^{i\vec{k}_0 \cdot \vec{R}_A} \phi_0^B(r_b)$$

$$E_{in} = \frac{P_B^2}{2m_B} + \frac{P_0^2}{2m_A} + E_p^B + E_0^A$$



k_p and k_0 denote the momentum of the atom A and B while $P_0^2 = k_p \cdot k_p$ and $P_0^2 = k_0 \cdot k_0$. In the final state

$$\psi_n = e^{i\vec{k}_g \cdot \vec{R}_B} \phi_g^B(r_b) e^{i\vec{k}_i \cdot \vec{R}_A} \phi_i^A(r_a)$$

$$E_n = \frac{k_g^2}{2m_B} + \frac{k_i^2}{2m_A} + E_g^B + E_i^A$$

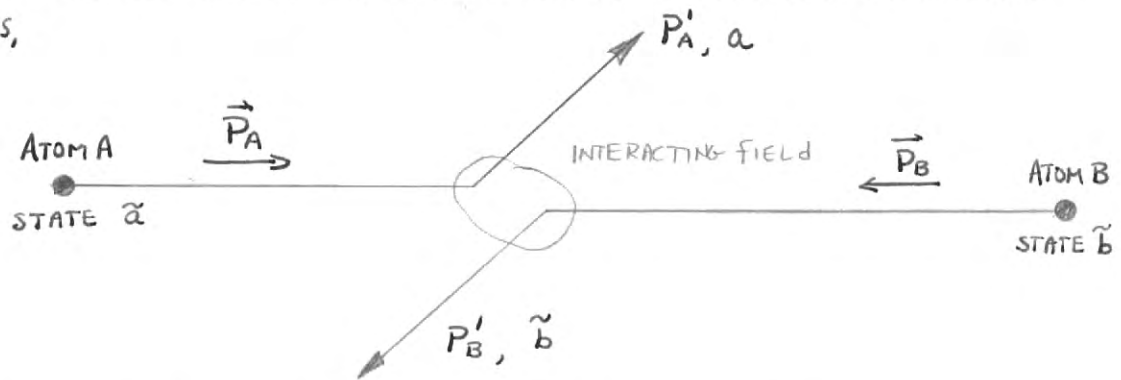
In this interaction it is possible to separate the translational motion of the center of mass from the internal motions of the atoms so that

$$\langle H_{in} \rangle = \int e^{i\vec{k}_g \cdot \vec{R}_B} e^{i\vec{k}_i \cdot \vec{R}_A} \phi_g^B(r_b) \phi_i^A(r_a) \frac{e^2}{|\vec{r}_A + \vec{R}_A - \vec{r}_B - \vec{R}_B|} e^{i\vec{k}_p \cdot \vec{R}_B} e^{i\vec{k}_0 \cdot \vec{R}_A} \phi_p^A \phi_0^B d^3r_a d^3r_b d^3R$$

40. TRANSITION RATES AND SCATTERING THEORY

I'D LIKE TO RETURN TO THE PROBLEM THAT I STARTED LAST TIME SINCE I DIDN'T GET IT FINISHED. I WAS DISCUSSING, WHAT TURNS OUT TO BE, A COMPLICATED INTERACTION. I HAD TWO PARTICLES COLLIDING AND IN THE PROCESS OF SCATTERING THEY EXCHANGE ENERGY SO ONE ATOM IS NO LONGER ~~IONIZED~~ ~~WA~~ EXCITED WHILE THE OTHER ATOM BECOMES IONIZED. BECAUSE I DIDN'T GET FAR I'LL START ALL OVER AGAIN.

I HAVE TWO ATOMS A AND B COMING TOWARDS EACHOTHER. ATOM A HAS MOMENTUM P_A WHILE BEING IN AN EXCITED STATE \tilde{a} . AFTER THE COLLISION IT HAS MOMENTUM P'_A WHILE MAKING A TRANSITION TO THE GROUND STATE. ATOM B, COMING TOWARDS A AT MOMENTUM P_B , COLLIDES WITH A AND SCATTERS OFF WITH MOMENTUM P'_B AND IN AN IONIZED STATE. DIAGRAMMATICALLY WE HAVE THE FOLLOWING PROCESS,



TO SUMMARIZES THEM THE CONDITIONS OF THE PROBLEM

	ATOM	STATE	MOMENTUM	ENERGY
INITIAL CONDITIONS	A	\tilde{a}	P_A	E_A
	B	b	P_B	E_B
FINAL CONDITIONS	A	a	P'_A	E'_A
	B	\tilde{b}	P'_B	E'_B

IN ORDER TO SIMPLIFY THE PROBLEM INITIALLY I WILL NOT CONSIDER ANY INTERNAL EXCITATIONS OF THE NUCLEI. Therefore, I WILL LIMIT MY DISCUSSION HERE TO THE PHENOMENA OF ELASTIC SCATTERING. THE INELASTIC CASE IS STRAIGHT FORWARD BUT IT GETS MESSY DRAGGING ALONG ALL THE TERMS. IT'LL BE HARD ENOUGH KEEPING THE ELASTIC CASE STRAIGHT.

THE ANSWER TO THE PROBLEM IS CONTAINED IN THE GENERAL FORMULA,

$$\frac{P(i \rightarrow f)}{\Delta t} = \text{TRANSITION RATE} = \frac{2\pi}{\hbar} |H_{if}'|^2 \delta(E_f - E_i)$$

ALL WE HAVE TO DO IS EVALUATE THIS SIMPLE EXPRESSION! IN ORDER TO EVALUATE THE EQUATION WE NEED THE INITIAL WAVE FUNCTION OF THE SYSTEM. TO WRITE THE INITIAL WAVE FUNCTION I WILL MAKE USE OF THE THEOREM THAT THE INTERNAL MOTION OF THE ATOM CAN BE SEPARATED FROM IT'S CENTER OF MASS MOTION. THIS IS SUCH A USEFUL CONCEPT TO KNOW THAT I'D LIKE TO PROVE IT TO YOU.

SEPARABILITY OF INTERNAL AND CENTER OF MASS MOTIONS

CONSIDER A TWO PARTICLE SYSTEM WITH A HAMILTONIAN GIVEN BY

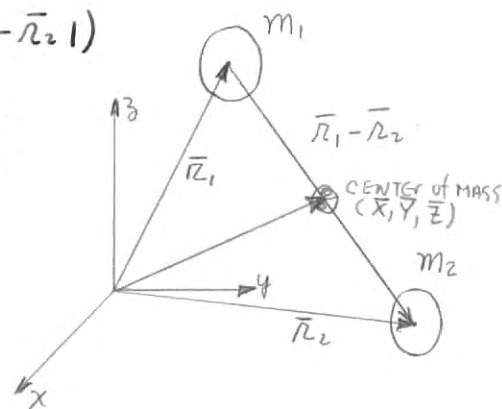
$$H = \frac{1}{2m_1} \nabla_{r_1}^2 + \frac{1}{2m_2} \nabla_{r_2}^2 + V(|\vec{r}_1 - \vec{r}_2|)$$

BECAUSE I'M LAZY AND BECAUSE OF THE SYMMETRY OF THE PROBLEM I'M ONLY GOING TO WORK WITH THE X-COMPONENTS. AND LATER ADD IN THE y AND z - TERMS.

I CAN FIRST DEFINE, IN THE CLASSICAL SENSE, THE X LOCATION OF THE CENTER OF MASS OF THE SYSTEM, I.E.,

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

I WILL CALL THE DISPLACEMENT, $x_1 - x_2$, ξ AND USE THE FOLLOWING RELATIONSHIP FROM THE THEORY OF DIFFERENTIATION.



If $x_1 - x_2 = \xi$ THEN

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1} \frac{\partial}{\partial \xi} + \frac{\partial \bar{x}}{\partial x_1} \frac{\partial}{\partial \bar{x}}$$

where $\bar{x} = \frac{m_1}{m_1+m_2} x_1 + \frac{m_2}{m_1+m_2} x_2 = \frac{m_1}{M} x_1 + \frac{m_2}{M} x_2$

AND $M = m_1 + m_2$. THEN

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi} + \frac{m_1}{M} \frac{\partial}{\partial \bar{x}}$$

SIMILARLY

$$\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial \xi} + \frac{m_2}{M} \frac{\partial}{\partial \bar{x}}$$

IN THE HAMILTONIAN THEN

$$\frac{1}{2m_1} \nabla_{r_1}^2 \rightarrow \frac{1}{2m_1} \left(\frac{\partial}{\partial x_1} \right)^2 = \frac{1}{2m_1} \left(\frac{\partial^2}{\partial \xi^2} + \frac{2m_1}{M} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{x}} + \frac{m_1^2}{M^2} \frac{\partial^2}{\partial \bar{x}^2} \right)$$

$$\frac{1}{2m_2} \nabla_{r_2}^2 \rightarrow \frac{1}{2m_2} \left(\frac{\partial}{\partial x_2} \right)^2 = \frac{1}{2m_2} \left(\frac{\partial^2}{\partial \xi^2} - \frac{2m_2}{M} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{x}} + \frac{m_2^2}{M^2} \frac{\partial^2}{\partial \bar{x}^2} \right)$$

AND $V(\bar{r}_1, \bar{r}_2) \rightarrow V(\xi)$

ADDING UP THE TERMS AND USING

$$y_1 - y_2 = \eta \quad \text{AND} \quad z_1 - z_2 = \delta$$

$$H = \left[\frac{1}{2m_1} \left(\frac{m_1}{M} \right)^2 + \frac{1}{2m_2} \left(\frac{m_2}{M} \right)^2 \right] \left[\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \delta^2} \right] + \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \delta^2} \right] + V(\xi, \eta, \delta)$$

THIS CAN BE REDUCED TO THE SIMPLER FORM

$$H = \frac{1}{2M} \nabla_{\bar{R}}^2 + \frac{1}{2\mu} \nabla_{\rho}^2 + V(\bar{\rho})$$

WHERE $\bar{\rho} = \xi \hat{i} + \eta \hat{j} + \delta \hat{k}$ IS THE RELATIVE COORDINATE BETWEEN ATOMS AND μ IS THE REDUCED MASS OF THE SYSTEM,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

\bar{R} DENOTES THE CENTER OF MASS LOCATION AND WE SEE THE HAMILTONIAN SPLITS INTO TWO DISTINCT PARTS

$$H = H_{\bar{R}} + H_{\bar{\rho}}$$

WHERE

$$H_{\bar{R}} = \frac{1}{2M} \nabla_{\bar{R}}^2 = \text{CENTER OF MASS MOTION}$$

$$H_{\bar{\rho}} = \frac{1}{2\mu} \nabla_{\rho}^2 + V(\bar{\rho}) = \text{INTERNAL INTERACTION.}$$

The reason we were successful in splitting up the motion into two parts came from adding up the two gradient terms $\nabla_{r_1}^2$ and $\nabla_{r_2}^2$ which contained cross terms which were equal but opposite in sign. Therefore the motions fortunately were uncoupled.

I should point out whenever one mass, say m_1 , is much greater than m_2 , then the reduced mass μ just becomes equivalent to the light mass m_2

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m_1 (1 + \frac{m_2}{m_1})} = m_2 (1 - \frac{m_2}{m_1} + \dots) \approx m_2 \text{ if } m_2 \ll m_1$$

Thus in the case of hydrogen where $m_1 =$ the neutron and $m_2 =$ the electron $\mu \approx$ mass of the electron since

$$\frac{m_N}{m_e} \approx 1800. \text{ But the fact that the reduced mass}$$

is the exact formulation of the Hamiltonian was recognized by Rutherford in the early days of quantum mechanics so we cannot ignore it in any precise calculations. End of side note!

ELASTIC SCATTERING TRANSITION RATES

Alright then, what is the wave function for the system initially and finally? We have established the fact that the Hamiltonian is separable into two parts: one corresponding to the center of mass motion, H_R , and the other part describes the internal motion, H_p . We know then

$$H = H_R + H_p = \frac{P^2}{2M} + E_i$$

where $\frac{P^2}{2M}$ is the energy due to the moving system and E_i are the energy levels of the system. The wave function is easily established by the product of the two wave functions

$$\psi \propto e^{i\vec{P}\cdot\vec{R}} \phi_i(\rho)$$

Here $e^{i\vec{P}\cdot\vec{R}}$ is the wave function describing the center of mass motion and $\phi_i(\rho)$ is the wave function describing the internal motion of the system.

SLOWLY THE PROBLEM IS TURNING INTO A REAL MORASS BECAUSE THE PROBLEM IS BECOMING OBSCURED BY THE COMPLEX NOTATION NEEDED TO KEEP EVERYTHING STRAIGHT. SO WE CONTINUE BY EXPRESSING EVERYTHING IN TERMS OF DIFFERENCE OF DISTANCES ρ_a, ρ_b , AND $\bar{\rho}_a - \bar{\rho}_b$. WE WILL HOLD OFF INTEGRATING OVER ρ_a AND ρ_b SINCE WE DON'T KNOW HOW TO NOR WILL WE BE ABLE TO BY MAKING APPROXIMATIONS; AS YET WE CAN EXPLAIN THE MACHINERY OF ATOM ENOUGH TO WRITE DOWN THE DYNAMICAL LAWS. IT IS SORT OF INTERESTING; QUANTUM MECHANICS IS NOTHING MORE THAN THE APPLICATION OF ONEUPSMANSHIP TO PHYSICS. WE USE THE CONCEPT WE KNOW BUT IN THE END WE NEVER REALLY CALCULATE ANYTHING - IT'S ALWAYS BEYOND OUR KNOWLEDGE.

WHAT WE WANT TO DO NOW IS TO REARRANGE THE INTEGRAL INTO FUNCTIONS OF THE SEPARATION DISTANCES. TO DO THAT WE REWORK THE EXPONENTIAL TERMS

$$e^{-i\bar{P}_a' \cdot \bar{\rho}_a} e^{-i\bar{P}_b' \cdot \bar{\rho}_b} e^{i\bar{P}_a \cdot \bar{\rho}_a} e^{i\bar{P}_b \cdot \bar{\rho}_b} = e^{-i\bar{P}_a' \cdot \bar{\rho}_a} e^{i\bar{P}_a' \cdot \bar{\rho}_b} e^{-i\bar{P}_a' \cdot \bar{\rho}_b} e^{-i\bar{P}_b' \cdot \bar{\rho}_b} e^{i\bar{P}_a \cdot \bar{\rho}_a} e^{-i\bar{P}_a \cdot \bar{\rho}_b} e^{i\bar{P}_a \cdot \bar{\rho}_b} e^{i\bar{P}_b \cdot \bar{\rho}_b}$$

WE CAN REARRANGE THIS NOW

$$e^{-i\bar{P}_a' (\bar{\rho}_a - \bar{\rho}_b)} e^{-i(\bar{P}_a' + \bar{P}_b') \cdot \bar{\rho}_b} e^{i\bar{P}_a (\bar{\rho}_a - \bar{\rho}_b)} e^{i(\bar{P}_a + \bar{P}_b) \cdot \bar{\rho}_b}$$

IF WE INTEGRATE THIS FUNCTION OVER $\bar{\rho}_b$ WE GET THE INTERESTING RESULT THAT

$$\int e^{+i[(\bar{P}_a + \bar{P}_b) - (\bar{P}_a' + \bar{P}_b')] \cdot \bar{\rho}_b} d\bar{\rho}_b = \begin{cases} 0 & \text{if } \bar{P}_a + \bar{P}_b \neq \bar{P}_a' + \bar{P}_b' \\ 1 & \text{if } \bar{P}_a + \bar{P}_b = \bar{P}_a' + \bar{P}_b' \end{cases}$$

WHICH TELLS US THAT THE PROBLEM WILL ONLY WORK OUT IF YOU CONSERVE MOMENTUM DURING THE INTERACTION. SO IF YOU DIDN'T MAKE THE INITIAL STATEMENT ABOUT CONSERVING MOMENTUM, THE EQUATIONS MAKE IT FOR YOU. IT'S INTERESTING; YOU CAN BE DUMB AND NATURE TAKES CARE OF YOU!

WE CAN NOW WRITE THE INITIAL AND FINAL WAVEFUNCTIONS

$$\text{INITIALLY } \psi_i = e^{i\vec{P}_a \cdot \vec{R}_a} \varphi_a(\rho_a) e^{i\vec{P}_b \cdot \vec{R}_b} \varphi_b(\rho_b)$$

$$\text{FINALLY } \psi_f = e^{i\vec{P}'_a \cdot \vec{R}_a} \varphi_a(\rho_a) e^{i\vec{P}'_b \cdot \vec{R}_b} \varphi_b(\rho_b)$$

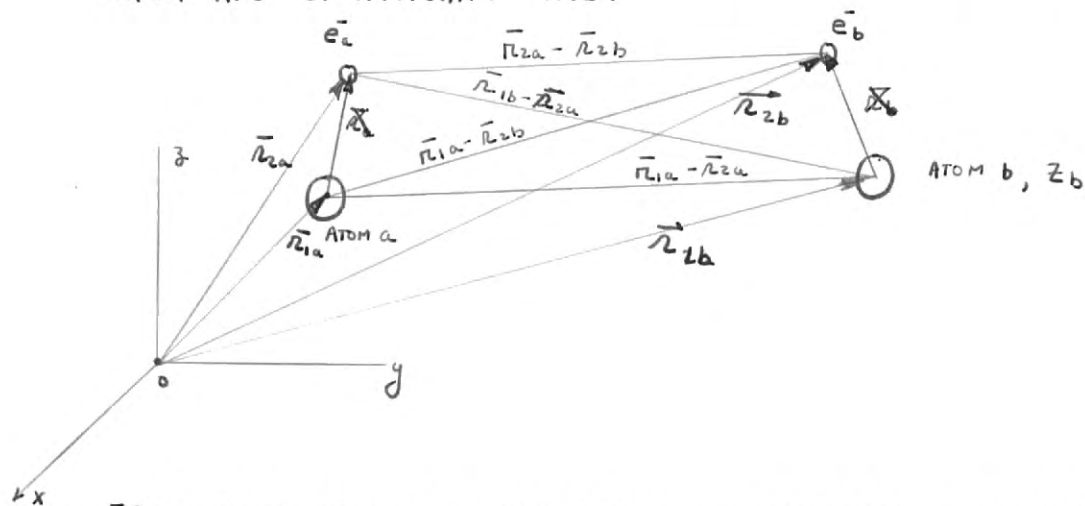
WE NOW MUST EVALUATE THE INTEGRAL,

$$\langle \psi_f | H | \psi_i \rangle = \int \psi_f^* V \psi_i dVOL.$$

SO LET'S WRITE IT OUT,

$$\int = \int d^3\vec{r}_a d^3\vec{r}_b d^3\vec{r}'_a d^3\vec{r}'_b \left[e^{-i\vec{P}'_a \cdot \vec{R}_a} \varphi_a^*(\rho_a) e^{-i\vec{P}'_b \cdot \vec{R}_b} \varphi_b^*(\rho_b) V(\vec{r}_a, \vec{r}_b, \vec{r}_a - \vec{r}_b) \right] e^{i\vec{P}_a \cdot \vec{R}_a} \varphi_a(\rho_a) e^{i\vec{P}_b \cdot \vec{R}_b} \varphi_b(\rho_b)$$

THIS LOOKS TERRIBLY CONFUSING BUT IF YOU FOLLOW THROUGH IT YOU MIGHT UNDERSTAND IT. TO CONTINUE WE'LL HAVE TO MAKE SOME APPROXIMATIONS ON THE POTENTIAL FUNCTION V . THE CO-ORDINATES WHICH ARE SIGNIFICANT ARE:



FROM THIS DIAGRAM AND WITH SOME PATIENCE WE CAN WRITE DOWN THE APPROPRIATE POTENTIAL FUNCTION

$$V = \frac{e^2}{|\vec{r}_{za} - \vec{r}_{zb}|} - \frac{Z_a e^2}{|\vec{r}_{za} - \vec{r}_{zb}|} - \frac{Z_b e^2}{|\vec{r}_{zb} - \vec{r}_{za}|} + \frac{Z_a Z_b e^2}{|\vec{r}_{za} - \vec{r}_{za}|}$$

NOW THAT WE HAVE AN EXPRESSION FOR V WE SEE THAT IT IS NOT IN THE FORM WE WANT IT SO THAT WE CAN PERFORM THE INTEGRATION; WE NEED $V(\rho_a, \rho_b, \vec{r}_a - \vec{r}_b)$. SO WE HAVE TO RE-EXPRESS V IN TERMS OF THE RIGHT VARIABLES.

TRANSITION RATE, SCATTERING CROSS-SECTION, AND ELASTIC SCATTERING

The PROBLEM HAS NOW BEEN PUT INTO THE GENERAL FORM WHICH IS READY TO BE INTEGRATED

$$\langle \Psi_f | H_{if} | \Psi_i \rangle = \int e^{-i\vec{p}_a' \cdot (\vec{r}_a - \vec{r}_b)} \phi_a^* \phi_b^* V e^{-i\vec{p}_a \cdot (\vec{r}_a - \vec{r}_b)} \phi_a \phi_b d^3r_a d^3r_b$$

SUBJECT TO THE CONSTRAINT THAT $p_a + p_b = p_a' + p_b'$. IN ORDER TO GO ON FROM HERE IT IS NECESSARY TO MAKE SOME APPROXIMATIONS.

BEFORE I GO ON I WANT TO MAKE USE OF THE SCATTERING CROSS SECTION WHICH WE INTRODUCED A WHILE BACK. IT CAN BE SHOWN THAT THE CROSS SECTION IS RELATED TO THE TRANSITION RATES BY THE RELATIVE COLLISION VELOCITY, I.E.,

$$\frac{dP}{dt} = v\sigma$$

Thus

$$\sigma = \frac{2\pi}{\hbar v} |H_{if}|^2 \delta(E_f - E_i)$$

LET'S TRY TO WORK OUT AN EASY PROBLEM WITH ELASTIC SCATTERING. BY ASSUMING UNEXCITED INTERNAL MOTION WE CAN PERFORM THE INTEGRATION.

ASSUME THEN LOW ENERGY ELASTIC SCATTERING AND THE ATOM REMAIN IN THE GROUND STATE. WE HAVE NOW CHANGED THE PROBLEM AND ARE NO LONGER WORKING WITH TRANSITIONS OF ONE EXCITED STATE TO A GROUND STATE. THUS THE MATRIX ELEMENT BECOMES

$$\langle \Psi_f | H_{if} | \Psi_i \rangle = \left[\iint V(|\vec{r}_a - \vec{r}_b|, \rho_a, \rho_b) |\phi_a^*(\rho_a)|^2 |\phi_b(\rho_b)|^2 d^3\rho_a d^3\rho_b \right] \left[\int e^{i(\vec{p}_a - \vec{p}_a') \cdot (\vec{r}_a - \vec{r}_b)} d^3(\vec{r}_a - \vec{r}_b) \right]$$

BECAUSE WE CAN'T DO THE DOUBLE INTEGRAL IN THE FIRST BRACKETS WE'LL JUST CALL IT $U(|\vec{r}_a - \vec{r}_b|)$, THE AVERAGE ELECTRICAL POTENTIAL. IT'S AN AVERAGE OF THE POSITION OF THE CHARGES WITHIN THE ATOMS. IF YOU ASSUME THAT THE CHARGE IS SPHERICALLY DISTRIBUTED OR ASSUME SOME OTHER INTEGRABLE MODEL, THEN YOU CAN EXPLICITLY EVALUATE IT. OTHERWISE YOU DON'T WORRY ABOUT IT AND ESTABLISH IT THROUGH EXPERIMENTS.

BY THE PREVIOUS ASSUMPTIONS WE NOW HAVE REDUCED THE PROBLEM CONSIDERABLY SINCE NOW WE MUST INTEGRATE,

$$\langle \psi_f | H_{if} | \psi_i \rangle = \int d^3(\bar{r}_a - \bar{r}_b) U(\bar{r}_a - \bar{r}_b) e^{i(\bar{p}_a - \bar{p}_a') \cdot (\bar{r}_a - \bar{r}_b)}$$

BY TRANSFORMING TO A NEW COORDINATE $\mathbf{r} = \bar{r}_a - \bar{r}_b$ WE CAN USE FOURIER TRANSFORM THEORY TO INTEGRATE

$$H_{if} = \int d^3 r U(\mathbf{r}) e^{i\mathbf{Q} \cdot \mathbf{r}}$$

WHERE $\mathbf{Q} = \bar{p}_a - \bar{p}_a'$; NOTICE THAT THE TRANSFORMED MOMENTUM \mathbf{Q} ALSO EQUALS $\bar{p}_b' - \bar{p}_b$ DUE TO CONSERVATION OF MOMENTUM. TO SHOW YOU HOW THIS INTEGRATION WORKS - LET'S ASSUME THAT

$$U = \frac{e^2}{z}$$

THEN

$$H_{if} = \int_0^\infty e^2 d^3 z \frac{e^{i\mathbf{Q} \cdot \mathbf{z}}}{z} = \frac{4\pi e^2}{Q^2}$$

THE SCATTERING CROSS-SECTION IS THEN

$$\sigma = \frac{2\pi}{\hbar v} \left(\frac{4\pi e^2}{Q^2} \right)^2 \delta(E_f - E_i)$$

NOW WE HAVE TO FIGURE OUT WHAT TO DO WITH THE DELTA FUNCTION. WHAT WE ARE INTERESTED IN DOING IS SUMMING OVER THE DENSITY OF STATES IN A GIVEN MOMENTUM RANGE; WE CAN LOOK FOR EITHER \bar{p}_a' OR \bar{p}_b' . WE MUST SUM OVER THE DENSITY OF STATES IN RANGE $d^3 \bar{p}_a'$, I.E.,

$$\sigma = \frac{2\pi}{\hbar v} \left(\frac{16\pi^2 e^4}{Q^4} \right) \delta(E_f - E_i) \frac{d^3 \bar{p}_a'}{(2\pi)^3}$$

THE ENERGY DIFFERENCE $E_f - E_i$ CAN BE EVALUATED

$$E_f - E_i = \frac{(\bar{p}_a')^2}{2m_a} + \frac{(\bar{p}_b')^2}{2m_b} - \frac{(\bar{p}_a)^2}{2m_a} - \frac{\bar{p}_b^2}{2m_b}$$

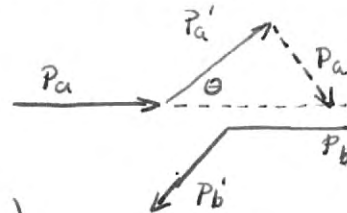
SINCE WE ARE SUMMING OVER \bar{p}_a' WE MUST GET RID OF \bar{p}_b' BY USING THE CONSERVATION OF MOMENTUM

$$(\bar{p}_b')^2 = (\bar{p}_a + \bar{p}_b - \bar{p}_a')^2$$

IF WE ASSUME THE INTEGRATION OVER $d^3 \bar{p}_a'$ IS SPHERICALLY SYMMETRICAL THE $d^3 \bar{p}_a' \rightarrow 4\pi (\bar{p}_a')^2 d\bar{p}_a'$ AND THE INTEGRATION WITH THE DELTA FUNCTION IS STRAIGHT FORWARD

If we have the special case where $P_b = P_b'$; i.e., the B atom comes in and leaves with the same momentum while scattering, then

$$Q = P_a - P_a' = 2 P_a \sin \frac{\theta}{2}$$



And

$$\sigma = \frac{2\pi}{h\nu} \times \frac{16\pi^2 e^4}{Q^4} \int \frac{(P_a')^2 dP_a' d\Omega_a'}{(2\pi)^3} \delta\left(\frac{P_a'^2}{2ma} - \frac{P_a^2}{2ma}\right)$$

To integrate we need to use the delta function property:

$$\delta f(x) dx = \frac{1}{|f'(x)|}$$

$$\sigma = \frac{4e^4 P_a' d\Omega_a'}{2h\nu ma Q^4}$$

where $d\Omega_a'$ is the element of solid angle for the scattered a particle. Since $Q = 2 P_a \sin \frac{\theta}{2}$

$$\sigma = \frac{1}{2} \frac{e^2 P_a'}{h\nu ma P_a^4} \frac{d\Omega_a'}{\sin^4 \frac{\theta}{2}}$$

Here then is the familiar scattering law which depends on $1/\sin^4 \frac{\theta}{2}$ as the scattering angular dependence.

QED. Supplement

41. INTERACTION OF A RADIATION FIELD AND MATTER

- SPONTANEOUS EMISSION FROM AN ATOM IN A QUANTIZED FIELD -

I'd now like to talk about the first real problem in quantum electrodynamics. The problem I want to work out is to predict with what probability an atom will emit a photon, excite the radiation field, and emit "light." To work this problem we have to go beyond the classical theory of light scattering from atoms and quantize the radiation field. We shall consider a single electron system, i.e., a hydrogen atom because there aren't too many things to keep track of. This is one of the real problems of QED; keeping straight the notation is a major task. I hope to make the presentation as clear as possible.

We will consider the atom to initially be in a state of energy E_i and see what happens as it makes a transition to state E_0 . The transition will take place in the "dark." By "dark" I mean that the background radiation is in its ground state; i.e. there are no photons in any modes. The energy in each mode of the field is the ground state energy $\hbar\omega/2$. We have pulled the first slight of hand which characterizes QED by redefining the vacuum state to be the zero energy state. We do this because if we sum over all modes, the zero point energy is infinite. To make the energy finite it is necessary to cutoff of the frequency range in an artificial way. This is the first symptom of internal difficulties with QED.

We will take the atom to be in an enormous box which contains many modes of the radiation field. The box is large so we don't have to worry about reflections from the sides. Each mode in the field will be modeled as a classical harmonic oscillator. Each mode will be characterized by two indices, k and i . k denotes the wave number of the mode while i represents the state of polarization which is either up or down so i has 2 values only. The displacement of the k^{th} mode is given by coordinate $q_{k,i}$. We will frequently be concerned

WILL THE NUMBER OF MODES PER UNIT VOLUME WHICH IS $\frac{d^3k}{(2\pi)^3}$.
 THE COORDINATE $g_{k,i}$ IS IMPORTANT BECAUSE IT CHARACTERIZES THE STRENGTH OF THE VECTOR POTENTIAL IN THAT MODE. THE VECTOR POTENTIAL OF THE FIELD IS DESCRIBABLE IN THE FOLLOWING FORM

$$\vec{A} = \sum_{k,i=1,2} \frac{\vec{e}_i}{\sqrt{2\omega_k}} g_{k,i} e^{i\vec{k}\cdot\vec{x}}$$

\vec{e}_i IS A UNIT VECTOR IN THE DIRECTION OF THE POLARIZATION OF THE FIELD. THE DENOMINATOR $\sqrt{2\omega_k}$ IS A NORMALIZATION FACTOR WHICH IS RELATED TO k SINCE $\omega_k^2 = c^2 k^2$. \vec{e} IS ORTHOGONAL TO \vec{k} , I.E. $\vec{k} \cdot \vec{e}_{k,i} = 0$.

NOW WE HAVE TO WRITE DOWN THE HAMILTONIAN OF THE SYSTEM. THERE WILL BE THREE PARTS ONE DUE TO THE ATOM BY ITSELF, THE SECOND PART DUE TO THE FIELD OSCILLATORS AND THE THIRD PIECE IS THE INTERACTION PART, I.E.,

$$H = H_{\text{ATOM}} + H_{\text{OSCIL.}} + H_{\text{INTER.}}$$

THE OSCILLATOR TERM CAN BE WRITTEN OUT IN MORE DETAIL AS

$$H_{\text{osc}} = \sum_{k,i} \frac{1}{2} (p_{k,i}^2 + \omega_{k,i}^2 q_{k,i}^2)$$

IT IS BEST TO CHANGE OVER TO OPERATOR NOTATION FOR WORKING WITH THIS HAMILTONIAN. RECALL OUR SOLUTION TO THE HARMONIC OSCILLATOR BACK ON PAGE 185. WE DEFINED THE OPERATORS,

$$q_{k,i} = \sqrt{\frac{2}{m\omega_{k,i}}} (a_{k,i} + a_{k,i}^*)$$

AND

$$p_{k,i} = i \sqrt{\frac{m\omega_{k,i}}{2}} (a_{k,i}^* - a_{k,i})$$

a AND a^* WERE REFERRED TO CREATION AND ANNIHILATION OPERATORS IN THAT THEY WORKED ON THE WAVEFUNCTION OF THE OSCILLATOR TO EITHER RAISE ITS ENERGY BY ONE UNIT OF $\hbar\omega$ (ONE PHOTON) OR REDUCE THE ENERGY BY ONE $\hbar\omega$ (EMIT ONE PHOTON). THE OPERATORS ARE ALSO SUBJECT TO THE COMMUTATOR RELATIONSHIP

$$aa^* - a^*a = 1$$

USING THE OPERATOR NOTATION JUST DESCRIBED, WE CAN WRITE THE HAMILTONIAN OF THE OSCILLATOR AS

$$H_{osc} = \sum \hbar \omega_{k,i} a_{k,i}^* a_{k,i}$$

THE VECTOR POTENTIAL MUST ALSO BE EXPRESSED IN TERMS OF a AND a^* . TO DO THAT JUST SUBSTITUTE $\phi_{k,i}$ INTO A ,

$$A = \sum_{k,i} \left[\frac{\vec{e}_{k,i}^*}{\sqrt{2\hbar\omega_{k,i}}} a_{k,i}^* e^{-i\vec{k}\cdot\vec{x}} + \frac{\vec{e}_{k,i}}{\sqrt{2\hbar\omega_{k,i}}} a_{k,i} e^{i\vec{k}\cdot\vec{x}} \right]$$

THE FIRST TERM IS THE CREATION TERM WHILE THE SECOND IS THE ANNIHILATION TERM. BY WE MEAN THE $a_{k,i}^*$ WILL CAUSE THE FIELD TO INCREASE FROM THE GROUND STATE BY ONE QUANTUM $\hbar\omega$ OR ONE PHOTON.

NOW WE HAVE TO THINK ABOUT HOW AN ELECTRON BEHAVES IN AN EXTERNAL FIELD. WE HAVE ALREADY SEEN THAT WITH AN INFINITELY HEAVY NUCLEUS WE CAN WRITE THE HAMILTONIAN FOR AN ATOM IN THE FIELD AS

$$H_0 = \frac{(\vec{P} - e\vec{A})^2}{2m} + V(r)$$

THE MASS IS THE ELECTRON MASS WHILE \vec{P} IS THE MOMENTUM OF THE ATOM. IF WE EXPAND THIS HAMILTONIAN OUT, WE WILL GET

$$H_0 = \frac{p^2}{2m} - \frac{e}{2mc} [\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}] + \frac{e^2}{2mc^2} \vec{A}\cdot\vec{A} + V(r)$$

NOW WE CAN SINGLE OUT THE THREE TERMS IN THE TOTAL HAMILTONIAN $H = H_{osc} + H_{atom} + H_{int}$

$$H_{osc} = \sum \hbar \omega_{k,i} a^* a \quad , \quad \text{behavior of field without an atom to interact with}$$

$$H_{atom} = \frac{p^2}{2m} + V(r) \quad , \quad \text{hydrogen hamiltonian without a field to interact with}$$

$$H_{int} = -\frac{e}{2mc} [\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}] + \frac{e^2}{2mc^2} \vec{A}\cdot\vec{A} \quad , \quad \text{interaction terms between field and atom}$$

THE SUM OF THE PIECES GIVES THE TOTAL COMPLETE DESCRIPTION OF THE SYSTEM.

WHAT HAS SO FAR APPEARED TO BE QUITE SIMPLE AND STRAIGHT FORWARD IS, ON THE CONTRARY, QUITE SUBTLE. WHEN WE EXPRESSED $\vec{A}(x)$ IN TERMS OF THE FIELD OPERATORS a AND a^* , WE HAVE INTRODUCED A LOT OF TECHNICAL DIFFICULTIES. SUCH PROBLEMS AS INFINITIES CROP UP WHICH HAVE TO BE DEALT WITH BY TRICKS AND MAGIC AND A LOT OF HAND WAVING. AS WE UNDERSTAND Q.E.D TODAY, THERE IS ~~AN~~ ONLY ONE FORMULATION OF WHICH RESULTS FROM MODELING THE FIELD AS A SERIES OF OSCILLATORS. IT IS A CONSEQUENCE OF THIS FORMULATION THAT IT LEADS TO THE TECHNICAL DIFFICULTIES WHICH CAUSE US TO DOUBT THE VALIDITY OF Q.E.D. WE SHALL DISCUSS THESE PROBLEMS AS THEY COME UP. SO LET'S PROCEED AND SEE WHAT IS IN STORE FOR US.

WE WILL BEGIN BY EXAMINING THE INTERACTION PIECE OF THE HAMILTONIAN WHICH CAUSES ALL THE DIFFICULTY. SINCE WE ARE GOING TO RESTRICT OUR PROBLEM TO FIRST ORDER PERTURBATION EFFECTS IT IS POSSIBLE TO DISCARD THE SECOND ORDER TERM IN \vec{A} , I.E. THE A^2 TERM. ALSO IF WE CONSIDER THE GAGE $\vec{\nabla} \cdot \vec{A} = 0$ THEN $\vec{p} \cdot \vec{A} = \vec{A} \cdot \vec{p}$ AND,

$$H_{INT} = -\frac{e}{mc} \vec{p} \cdot \vec{A}(x)$$

\vec{p} IS THE MOMENTUM OF ELECTRON AND THE VECTOR POTENTIAL IS DEFINED AT THE POSITION OF THE ELECTRON. NOW SUBSTITUTING FOR \vec{A} WE HAVE

$$H_{INT} = -\frac{e}{mc} \sum_{k,i} \left[\frac{(\vec{p} \cdot \vec{e}_{k,i}^*)}{\sqrt{2\omega_{k,i}}} e^{i(k \cdot x - \omega t)} a_{k,i}^* + \frac{(\vec{p} \cdot \vec{e}_{k,i})}{\sqrt{2\omega_{k,i}}} e^{i(k \cdot x - \omega t)} a_{k,i} \right]$$

NOW WE ARE READY TO COMPUTE THE TRANSITION RATE,

$$\frac{\text{Prob}(i \rightarrow f)}{\text{SEC}} = \frac{2\pi}{\hbar} |H_{INT}(f,i)|^2 \delta(E_f - E_i)$$

BUT TO EVALUATE THIS EXPRESSION WE MUST COMPUTE THE MATRIX ELEMENT $\langle \psi_f | H_{INT} | \psi_i \rangle$ AND THEN SQUARE IT TO GET THE PROBABILITY. OKAY, WE NEED TO DESCRIBE THE INITIAL AND FINAL WAVE FUNCTION OF THE SYSTEM.

THE INITIAL WAVE FUNCTION IS GIVEN AS

$$\psi_i = |A_1; 0, 0, 0, \dots, 0, 0, 0\rangle$$

IN WORDS THIS SAYS THAT THE ATOM IS IN STATE 1 (AN EXCITED STATE) WHILE THE FIELD IS IN THE GROUND STATE. THUS ALL THE ZEROES (AN INFINITE NUMBER OF THEM) DESCRIBE THE ENERGY CONTENT OF EACH MODE. THE FINAL STATE OF THE SYSTEM IS

$$\psi_f = |A_0; 0, 0, 0, 1, \dots, 0, 0, 0\rangle$$

THIS SAYS THE ATOM IS NOW IN ITS GROUND STATE WHILE THE FIELD HAS ONE MODE WHICH HAS ABSORBED ONE PHOTON OF ENERGY, - THE ENERGY THE ATOM LOST. WE'LL CHARACTERIZE THE EXCITED MODE AS $k = l$ AND $i = j$. THE ENERGY STATES OF THE SYSTEM ARE GIVEN AS

<u>CONDITION</u>	<u>ENERGY</u>	<u>ATOM</u>	<u>FIELD</u>
INITIALLY	$E_i = E_1 + 0$	$= E_1$	$+ 0$
FINALLY	$E_f = E_0 + \hbar \omega_l = E_0 + \hbar c l$		

LET'S WRITE OUT ψ_i AND ψ_f IN TERMS OF THE PRODUCT OF WAVE FUNCTION TO HELP CLARIFY THE MEANING OF THE DIRAC NOTATION AS USED ABOVE.

$$\psi_i = |A_1; 0, 0, 0, \dots, 0, 0, 0\rangle = \Phi_1(x_0) \phi_0(q_1) \phi_0(q_2) \phi_0(q_3) \dots \phi_0(q_n)$$

$\Phi_1(x_0)$ IS THE WAVE FUNCTION OF THE ATOM IN THE FIRST EXCITED STATE. THE MATHEMATICAL EXPRESSION OF $\Phi_1(x_0)$ IS x TIMES A GAUSSIAN DISTRIBUTION. THE WAVE FUNCTION OF THE N^{th} MODE OF THE FIELD IS GIVEN BY A GAUSSIAN DISTRIBUTION SINCE IT IS IN THE GROUND STATE (ALL MODES HAVE GAUSSIAN DISTRIBUTIONS INITIALLY THEREFORE $\phi_0(q_{k,i,n}) \propto e^{-\frac{1}{2} \omega_{ki}^2 q_n^2}$). IN THE FINAL STATE THE SYSTEM WAVE FUNCTION IS GIVEN BY

$$\psi_f = |A_0; 0, 0, 0, 1, \dots, 0, 0, 0\rangle = \Phi_0(x_0) \phi_0(q_1) \phi_0(q_2) \phi_0(q_3) \phi_1(q_4) \dots \phi_0(q_n)$$

HERE THE ATOM HAS A WAVE FUNCTION $\Phi_0(x_0)$ WHICH IS A GAUSSIAN SINCE IT IS IN THE GROUND STATE. THE FIELD NOW HAS ONE MODE WHICH IS EXCITED, CALL IT THE l, j MODE WHICH HAS A WAVE FUNCTION $\phi_1(q_{l,j,4})$. NOW WE CAN TRY TO COMPUTE THE RATE OF TRANSITION WITH THE INFORMATION AVAILABLE TO USE.

The MATRIX ELEMENT MUST BE WORKED OUT, I.E.,

$$(H_{\text{INT}})_{f,i} = \langle A_0; 0,0,0,1, \dots, 0,0,0 | H_{\text{INT}} | A_1; 0,0,0, \dots, 0,0,0 \rangle$$

NOW SUBSTITUTE H_{INT} INTO THIS EXPRESSION,

$$(H_{\text{INT}})_{f,i} = -\frac{e}{mc} \sum_{\mathbf{k},i} \langle A_0; 0,0,0,1, \dots, 0,0,0 | \left(\frac{\vec{p} \cdot \vec{e}_{\mathbf{k},i}^*}{\sqrt{2\omega_{\mathbf{k},i}}} e^{-i\mathbf{k} \cdot \vec{r}} a_{\mathbf{k},i}^* + \frac{\vec{p} \cdot \vec{e}_{\mathbf{k},i}}{\sqrt{2\omega_{\mathbf{k},i}}} a_{\mathbf{k},i} e^{i\mathbf{k} \cdot \vec{r}} \right) | A_1; 0,0,0, \dots, 0,0,0 \rangle$$

WE HAVE TO FIGURE OUT WHAT THIS MESS REDUCES DOWN TO. IN ORDER TO PROCEED WE HAVE TO RECALL WHAT $a_{\mathbf{k},i}$ AND $a_{\mathbf{k},i}^*$ DO TO THE WAVE FUNCTIONS OF THE FIELD. FORTUNATELY WE ALREADY KNOW THE ANSWER; WE WORKED IT OUT BACK ON PAGE 137. a AND a^* HAVE THE PROPERTY THAT

$$a^* \phi_n = \sqrt{\frac{n+1}{2m}} \phi_{n+1}$$

$$a \phi_n = \sqrt{\frac{n}{2m}} \phi_{n-1}$$

THAT IS a^* CREATES A QUANTUM OF ENERGY AND THUS RAISE THE STATE OF THE SYSTEM ONE UNIT. a , ON THE OTHER HAND, LOWERS THE EIGENSTATE ONE STEP THUS CORRESPONDS TO EMISSION OF ONE PHOTON. WE SHOULD NOTE THAT THE OPERATION $a \phi_0$ PRODUCES ZERO SINCE ϕ_0 IS THE GROUND YOU CAN'T GO LOWER THAN THAT ONE. SIMILARLY THE PRODUCT $\phi_0 a^* = 0$ BY COMPLEX CONJUGATION RULES. THEREFORE WE DON'T HAVE TO WORRY ABOUT THE TERM $a | A_1; 0,0,0, \dots, 0,0,0 \rangle = 0$; THIS IS JUST A RESULT OF THE ABSENCE OF ANY PHOTONS INITIALLY WHICH CAN BE ANNIHILATED.

SO NOW WE SEE WHAT HAPPENS WHEN WE CALCULATE THE EFFECT OF $a_{\mathbf{k},i}^*$ ON $\langle A_0; 0,0,0,1, \dots, 0,0,0 \rangle$. SINCE $a_{\mathbf{k},i}^* \phi_0(g_n) = \sqrt{\frac{n+1}{2m}} \phi_{n+1}$ IT APPEARS THAT EACH MODE OF THE FIELD GETS EXCITED BY ONE PHOTON. BUT THAT'S NOT WHAT HAPPENS BECAUSE EACH PAIR OF MODES MUST BE ORTHOGONAL BETWEEN THE INITIAL AND FINAL STATES. SINCE IN THE FINAL STATE THERE IS ONE PHOTON IN THE \mathbb{L},j MODE, ALL THE OTHER \mathbf{k},i MODE MUST REMAIN ZERO OTHERWISE WE ARE PRODUCING MORE PHOTONS (I.E., ENERGY) THAN THE ATOM CAN SUPPLY. THE SUMMATION OVER ALL \mathbf{k},i IS ZERO UNLESS $\mathbf{k},i = \mathbb{L},j$. THEREFORE $a_{\mathbb{L},j}^* \phi_0(g_n) = \sqrt{\frac{1}{2m}} \phi_1(g_n)$

$$H_{f,i} = -\frac{e}{2mc\sqrt{\omega_{\mathbb{L},j}}} \langle A_0 | \vec{p} \cdot \vec{e}_{\mathbb{L},j}^* e^{-i\mathbb{L} \cdot \vec{r}} | A_1 \rangle$$

WE CAN NOW SQUARE THIS EXPRESSION TO COMPUTE THE TRANSITION RATE,

$$\frac{P(i \rightarrow f)}{\text{SEC}} = \frac{e^2}{4m^2c^2\omega_{ij}^2} \left| \langle A_0 | \vec{p} \cdot \vec{e}_{\omega, j}^* e^{-i\vec{k} \cdot \vec{x}} | A_1 \rangle \right|^2 \cdot \frac{2\pi}{\hbar} \delta(\epsilon_0 + \hbar\omega_{ij} - \epsilon_1)$$

HERE WE PUT IN THE INITIAL AND FINAL ENERGY VALUES IN THE DELTA FUNCTION. SO THIS IS THE RESULT BUT IN A FORM WHICH IS NOT TOO USEFUL. USUALLY WE ARE INTERESTED IN THE PROBABILITY OF FINDING THE EMITTED PHOTON IN SOME DIRECTION IN A SOLID ANGLE $d\Omega$. WE WANT TO INTEGRATE THE ABOVE EXPRESSION OVER THE VOLUME $\frac{d^3\mathbb{L}}{(2\pi)^3}$ OR IN TERMS OF SOLID ANGLES OVER THE DIFFERENTIAL VOLUME, $\frac{d^3\mathbb{L}}{(2\pi)^3}$

$$\frac{d^3\mathbb{L}}{(2\pi)^3} \rightarrow \frac{\mathbb{L}^2 d\mathbb{L} d\Omega}{(2\pi)^3}$$

Thus

$$\frac{P(i \rightarrow f)}{\text{SEC}} = \frac{e^2}{4m^2c^2\omega_{ij}^2} \left| \langle A_0 | \vec{p} \cdot \vec{e}_{\omega, j}^* e^{-i\mathbb{L} \cdot \vec{x}} | A_1 \rangle \right|^2 \cdot \frac{2\pi}{\hbar} \int \delta(\epsilon_0 - \epsilon_1 + \hbar c\mathbb{L}) \frac{\mathbb{L}^2 d\mathbb{L} d\Omega}{(2\pi)^3}$$

HERE I SUBSTITUTED $\omega_{ij} = \mathbb{L}c$ INTO THE DELTA FUNCTION TO GET THE RIGHT FORM FOR INTEGRATION. IT IS NOW STRAIGHT FORWARD FROM HERE SINCE $\delta(f(x)) dx = \frac{1}{f'(x)} = \frac{1}{c\hbar}$

$$\frac{P(i \rightarrow f)}{\text{SEC}} = \frac{e^2}{16\pi^2 m^2 c^2 \omega_{ij}^2 \pi^2 \hbar^2} \left| \langle A_0 | \vec{p} \cdot \vec{e}_{\omega, j}^* e^{-i\mathbb{L} \cdot \vec{x}} | A_1 \rangle \right|^2$$

WE CAN SIMPLIFY THIS SOMEWHAT SINCE $\frac{\mathbb{L}^2}{\omega_{ij}^2} = \frac{1}{c^2}$

$$\frac{P(i \rightarrow f)}{\text{SEC}} \text{ INTO } d\Omega = \frac{e^2}{16\pi^2 m^2 c^5 \hbar^2} \left| \langle A_0 | \vec{p} \cdot \vec{e}_{\omega, j}^* e^{-i\mathbb{L} \cdot \vec{x}} | A_1 \rangle \right|^2 d\Omega$$

AND THIS IS THE ANSWER FOR CALCULATING THE RATE OF SPONTANEOUS EMISSION OF LIGHT AT FREQUENCY $\omega_{ij} = \frac{\epsilon_1 - \epsilon_0}{\hbar}$ INTO A DIRECTION $d\Omega$. THE ANSWER IS ALSO VALID \hbar FOR INDUCED ABSORPTION SINCE THE MATRIX ELEMENTS ARE EQUIVALENT.

42. REMARKS ON 1ST ORDER NON-RELATIVISTIC PERTURBATION THEORY

Before I go on to discuss the 2nd order perturbation theory and eventually discuss the relativistically invariant forms of QED, I'd like to discuss the limitations of the theory as we currently know it.

We have been dealing with problems where we want to calculate the probability per second that a system will make a transition from some initial state i to some final state f if the system is acted upon by an external potential. We learned the formulation of the transition rule (sometimes called Fermi's Golden Rule) as

$$\text{TRANS. PROB} \frac{\text{sec}}{\text{sec}} = \frac{2\pi}{\hbar} \left| \int_f (\text{POTENTIAL})_i \right|^2 \cdot (\text{DENSITY OF STATES})$$

where the density of states is given by

$$\frac{k^2 dk d\Omega}{(2\pi c)^3 dE} = \frac{\omega^2 d\Omega}{(2\pi c)^3 \hbar d\omega}$$

The potential could be a vector potential of the form

$$\vec{A} = \sum_{k,i} \sqrt{\frac{4\pi\hbar c^2}{2\omega_{k,i}}} \left[a_{k,i}^* e^{-i\mathbf{k}\cdot\mathbf{x}} \vec{e}_i + a_{k,i} e^{i\mathbf{k}\cdot\mathbf{x}} \vec{e}_i \right]$$

From this potential we developed the interaction Hamiltonian and wrote it as

$$H_{\text{int}} = \frac{e}{mc} \vec{p}_0 \cdot \vec{A}(x_0)$$

\vec{p}_0 being the electron momentum and $\vec{A}(x_0)$ is evaluated at the position of the electron. This interaction term was a consequence of expanding the total Hamiltonian of the system

$$H = \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 + eV$$

Now there is a piece of the Hamiltonian, i.e., $\frac{e^2}{2mc^2} \vec{A} \cdot \vec{A}$ which we have neglected. The reason we neglected it

WAS DUE TO THE APPROXIMATION MADE IN CALCULATING THE AMPLITUDE TO GO FROM i TO f , RECALL THAT

$$\langle \psi_f | \psi(t) \rangle = \langle \psi_f | \psi_i \rangle - \frac{i}{\hbar} \langle \psi_f | \int_0^T H_{\text{int}}(t) dt | \psi_i \rangle + \dots$$

BY EXPANDING TO FIRST ORDER WE ARE PHYSICALLY RESTRICTING THE POTENTIAL TO ACT ONLY ONCE BETWEEN i AND f . AS A CONSEQUENCE OF THIS FIRST ORDER APPROXIMATION, WE HAVE NO USE FOR THE $\bar{A} \cdot \bar{A}$ TERM. WHICH THIS TERM DESCRIBES A TWO-FOLD ACTION OF \bar{A} ON THE TRANSITION. SINCE WE ARE DEALING WITH FIRST ORDER APPROXIMATIONS THE OPERATORS a AND a^* ONLY WORK ON THE PHOTON STATE ONE TIME. THIS MEANS THAT THE FINAL AND INITIAL STATE CAN DIFFER BY ONLY ONE PHOTON. WHAT THEN OF A SYSTEM WHERE OUR CAVITY IS FILLED WITH A LOT OF PHOTONS IN SAY MODE μ, j (CALL THEM $n_{\mu, j}$ PHOTONS) AND A TRANSITION OCCURS BETWEEN i AND f ?

USING THE CREATION AND ANNIHILATION OPERATORS a^* AND a , RESPECTIVELY, WE SHOULD FIND IN GENERAL THAT THE PROBABILITY TO MAKE A TRANSITION IN THE DARK IS PROPORTIONAL TO ~~THE DARK~~ $n+1$. THIS FOLLOWS FROM THE QUANTUM THEORY OF OSCILLATORS WHICH PREDICTS

$$\langle n+1 | a^* | n \rangle = \sqrt{n+1} \langle n+1 | n \rangle$$

AND THE PROBABILITY TO ABSORB A PHOTON IS PROPORTIONAL TO n .

THIS FOLLOWS FROM

$$\langle n-1 | a | n \rangle = \sqrt{n} \langle n-1 | n \rangle$$

THE PROBABILITIES ARE THE SQUARE OF THESE MATRIX ELEMENTS. THE LAST EXAMPLE WE WORK OUT WAS FOR THE CASE $n=0$. THIS ASSUMED ALL THE MODES OF THE FIELD WERE IN THE GROUND STATE. THUS ONLY EMISSION COULD TAKE PLACE WHEREIN THE ATOM EXCITED THE FIELD WITH ONE PHOTON. WE CALLED THIS EMISSION SPONTANEOUS IN THE SENSE THAT IT DIDN'T NEED TO BE FORCED EXTERNALLY. THUS WITHOUT DOING TOO MUCH WORK WE HAVE LEARNED HOW TO EXTEND OUR RESULTS TO A SYSTEM WHERE BOTH ABSORPTION AND EMISSION CAN TAKE PLACE. FURTHER WE CAN CONCLUDE THAT:

$$\text{Probability of emission } i \rightarrow f = C_{if} (1 + n_k)$$

AND

$$\text{Probability of Adsorption } f \rightarrow i = B_{if} (n_k)$$

RADIATION EQUILIBRIUM

The coefficients $C \pm B$ are equal due to the fact that the complex conjugate of the vector potential equals the vector potential itself. From the results thus obtained we can ~~conclude~~ deduce something about thermal equilibrium. Suppose I have two states of a system, m and n , each containing N_m and N_n atoms per unit volume. From thermodynamics we saw that for equilibrium the two states must be in the ratio of

$$\frac{N_m}{N_n} = e^{-(E_m - E_n)/kT}$$

This system will be in equilibrium with the radiation field if the number of atoms going from m to n per unit time by absorption of photons ($\hbar\omega_{mn}$) equals the number going from n to m by emission. We must require then

$$N_n \eta_{mn} = N_m (1 + \eta_{mn})$$

or using the above relationship

$$\eta_{mn} = e^{-(E_m - E_n)/kT} (1 + \eta_{mn})$$

solving for η_{mn} ,

$$\eta_{mn} = \frac{1}{e^{(E_m - E_n)/kT} + 1} = \frac{1}{e^{\hbar\omega_{mn}/kT} + 1}$$

This is the Planck black-body distribution law for η_{mn} photons of frequency, ω_{mn} .

I might add one further comment that the emission process described here differs from the classical results, i.e., when the field is modeled as a classical wave, by the extra piece due to spontaneous emission. The results for absorption are consistent.

MANY ELECTRON SYSTEM

So far we have restricted our analysis to a single electron system, i.e., the hydrogen atom. In general we will have a interaction Hamiltonian which is the sum of all the individual electrons, i.e.,

$$H_{int.} = \sum_{\mu} \frac{e_{\mu}}{m_{\mu}c} \vec{p}_{\mu} \cdot \vec{E}_{k,i} e^{-i\mathbf{k} \cdot \mathbf{x}}$$

When dealing with many electrons at a time it is more convenient to define a new operator $\mathbf{J}(\mathbf{x})$, the current density,

$$\mathbf{J}(\mathbf{x}) = \sum_n \frac{e_n}{m_n c} p_n \delta^3(\mathbf{x} - \mathbf{x}_n)$$

The interaction term H_{INT} can be converted to an integral of the form

$$H_{INT} = \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}) d^3x$$

Substituting for $\mathbf{A}(\mathbf{x})$ we have

$$H_{INT} = \frac{1}{\sqrt{2\omega_k}} \int \left[\mathbf{J}(\mathbf{x}) \cdot \mathbf{e}_i a_{k,i} e^{-i\mathbf{k}\cdot\mathbf{x}} + \mathbf{J}(\mathbf{x}) \cdot \mathbf{e}_i a_{k,i} e^{i\mathbf{k}\cdot\mathbf{x}} \right] d^3x$$

Since $\mathbf{j}(\mathbf{k}) = \int \mathbf{J}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x =$ Fourier transform of $\mathbf{J}(\mathbf{x})$ we observe that the perturbation gets strong if there is a lot of juice ~~of~~ in the k^{th} mode. The amplitude to make emissions and absorptions is then:

$$\text{Amp. to emit} \propto \sqrt{n+1} |j(\mathbf{k})|_{mn} \bar{e}_{k,i}$$

$$\text{Amp. to absorb} \propto \sqrt{n} |j(\mathbf{k})|_{mn} \bar{e}_{k,i}$$

So far we have only considered time dependent perturbation theory as it applies to the Schrödinger equation. We had solved the wave equation

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

where H was expanded in a power series, $H_0 + H' + H'' + \dots$. A more accurate representation of the system is to make the Hamiltonian invariant under translation. This is done by using the Dirac equation, ~~equation~~

$$H_{D,0} = \alpha \cdot p + \beta m$$

where no external potentials are acting. If potentials are acting then

$$H_{D,A} = \alpha \cdot (p - \frac{e}{c} A) + \beta m + V$$

α and β are operators which work on the spin variables of the system. In the Dirac equation the interaction term becomes

$$H_{INT} = - \frac{e}{c} \alpha \cdot A$$

WE CAN WRITE THE HAMILTONIAN OF THE SYSTEM THEN AS

$$H = H_{0,A} + H_{osc}$$

AND SINCE $H_{0,A} = H_{0,0} + H_{INT}$ WE SEE THAT AGAIN THE HAMILTONIAN BREAKS IN 3 PARTS. FROM THIS EQUATION WE CAN GO ON TO DEVELOPE A MANIFESTLY RELATIVISTIC FORM FOR THE INTERACTION PROBLEM WITHOUT DOING A LOT OF TEDIOUS WORK. I'D LIKE FIRST TO WORK OUT SOME MORE PROBLEMS.

SCATTERING OF LIGHT FROM AN ATOM

SECOND ORDER TIME DEPENDENT PERTURBATION THEORY

WE ARE NOW READY TO GET MORE COMPLICATED BY CONSIDERING WHAT HAPPENS IN THE NEXT ORDER APPROXIMATION. LET'S REVIEW BRIEFLY WHERE WE ARE AND WHAT A 2ND PERTURBATION MEANS PHYSICALLY DURING A TRANSITION PROCESS. IMAGINE A TRANSITION TAKING PLACE IN THE TIME INTERVAL $0 \leq t \leq T$. WE WANT TO KNOW WITH WHAT PROBABILITY A SYSTEM, STARTING IN STATE i , WILL BE FOUND IN A PARTICULAR STATE f AFTER A TIME t . ONE WAY TO GET FROM $i \rightarrow f$ IS TO GO DIRECTLY WITHOUT ANY DISTURBANCE; WE CALL THIS THE ZERO ORDER TERM. THE AMPLITUDE TO GO FROM $i \rightarrow f$ WITHOUT AN INTERACTION

$$\text{IS JUST } \langle \psi_f | \psi_i \rangle = \delta_{fi} e^{-i(E_f - E_i)T} = \begin{cases} 1 & \text{if } E_f = E_i \\ 0 & \text{if } E_f \neq E_i \end{cases}$$

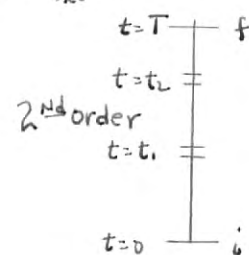
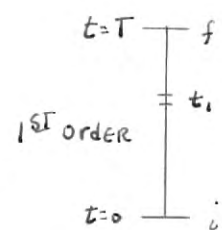
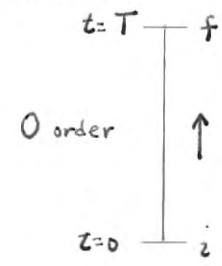
IN THE FIRST ORDER TERM THE INTERACTION OCCURS ONCE AT TIME $t = t_1$ AND THE AMPLITUDE TO ARRIVE AT f IS,

$$-\frac{i}{\hbar} \int_0^T dt \langle \psi_f | e^{+iE_f(T-t)} (H_{INT})_{fi} e^{+iE_i t} | \psi_i \rangle$$

THE SECOND ORDER TERM ALLOWS FOR TWO INTERACTIONS ONE AT t_1 AND THE OTHER AT t_2 . THE AMPLITUDE TO MAKE A TRANSITION IN SECOND ORDER IS

$$+ \sum_k \frac{(-i)^2}{\hbar^2} \int_0^T dt_2 \int_0^{t_2} dt_1 \langle \psi_f | e^{iE_f(T-t_2)} (H_{INT})_{fk} e^{+iE_k(t_2-t_1)} (H_{INT})_{ki} e^{iE_i t_1} | \psi_i \rangle$$

THE DIAGRAMS SHOW THE TIME ORDERING OF THE EVENTS. IN THE SECOND ORDER TERM $t_2 > t_1$ AND WE MUST SUM OVER ALL POSSIBLE INTERMEDIATE STATES k .



The INTERMEDIATE STATE IS OFTEN CALLED A VIRTUAL STATE SINCE IT ONLY LASTS A SHORT TIME. IN ORDER TO EVALUATE THE INTEGRAL WE HAVE TO AVERAGE OVER A LONG TIME T AS WE DID IN THE FIRST ORDER PROBLEM. TO PUT THE INTEGRAND INTO A MORE INTEGRABLE FORM CONSIDER THE FOLLOWING EXPONENTIAL RELATIONSHIP

$$e^{iE_f(T-t_2)} e^{iE_k(t_2-t_1)} e^{iE_i t_1} = e^{iE_i T} e^{i(E_f-E_i)(T-t_2)} e^{i(E_k-E_i)(t_2-t_1)}$$

WHEN T IS VERY LARGE $e^{iE_i T}$ AVERAGES OUT TO 1. THE MIDDLE TERM $e^{i(E_f-E_i)(T-t_2)}$ WILL PRODUCE THE DELTA FUNCTION, $\delta(E_f-E_i)$ SIMILAR TO LAST TIME. THE LAST TERM MUST BE INTEGRATED AND IS OF THE FORM

$$\int_0^\infty e^{i(E_k-E_i)t} dt$$

HERE WE HAVE AN INTEGRAL WHICH IS DIVERGENT AND OSCILLATES ABOUT ZERO UNLESS t_2-t_1 IS SMALL. WE FIRST INSERT A SMALL CONVERGING FACTOR, $e^{-\epsilon t}$, INTO THE INTEGRAL AND INTEGRATE.

IN GENERAL
$$\int_0^\infty e^{ix - \epsilon x} dx = \frac{1}{x + i\epsilon} = \text{P.V.} \frac{1}{x} - i\pi \delta(x)$$

Therefore
$$\int_0^\infty e^{i(E_k-E_i)t - \epsilon t} dt = \frac{i}{E_k-E_i + i\epsilon}$$

AND FINALLY SUMMING OVER ALL POSSIBLE STATES k , WE HAVE AS THE 2ND ORDER TRANSITION AMPLITUDE,

$$M_{fi}^{(2)} = - \sum_k \frac{i}{\hbar} \frac{(H_{int})_{fk} (H_{int})_{ki}}{E_k - E_i + i\epsilon}$$

THE MATRIX ELEMENTS $(H)_{fk}$ AND $(H)_{ki}$ ARE NEVER REALLY CALCULATED SO WE LEAVE THE ANSWER IN THIS FORM. WE CAN COMPARE THIS RESULT TO THE ZERO AND FIRST ORDER AMPLITUDE

$$M_{fi}^{(0)} = 1, \quad M_{fi}^{(1)} = -\frac{i}{\hbar} (H_{int})_{fi}$$

THE FORMULA FOR CALCULATING THE TRANSITION PROBABILITY CAN BE GENERALIZED IN THE FOLLOWING SIMPLE FORM:

$$\text{TRANS. PROB} \frac{\text{Sec}}{\text{Sec}} = \frac{2\pi}{\hbar} |M_{fi}|^2 \delta(E_f - E_i)$$

THIS IS A BEAUTIFULLY SIMPLE RESULT BECAUSE IT IS NOW POSSIBLE TO GO ON TO HIGHER ORDER TERMS WITHOUT DOING A LOT OF WORK. FOR EXAMPLE, THE THIRD ORDER TERM IS GIVEN BY

$$M_{fi}^{(3)} = \sum_r \sum_l \frac{(H_{int})_{fr} (H_{int})_{rl} (H_{int})_{li}}{(E_l - E_i + i\epsilon)(E_r - E_i + i\epsilon)}$$

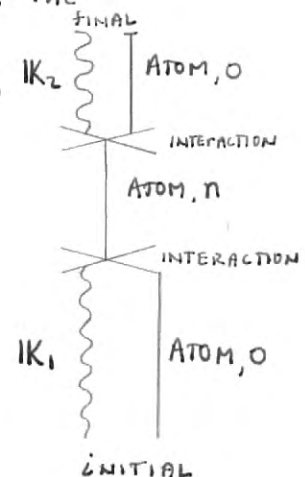
AT FIRST GLANCE YOU MAY THINK THE DENOMINATOR IS SCREWY AND UNSYMMETRIC AND SHOULD BE $(E_l - E_r + i\epsilon)(E_r - E_i + i\epsilon)$. THE IMPORTANT POINT TO REMEMBER IN WRITING DOWN THESE EXPRESSION IS THE STRENGTH OF THE INTERACTION (OR RESONANCE, IF YOU LIKE) DEPENDS ON HOW FAR YOU ARE AWAY FROM THE INITIAL STATE - ALWAYS! IF ALL THE EXPANSION TERMS ARE FIGHTING EACH OTHER AT THE SAME TIME, IT IS NECESSARY TO ADD UP ALL THE M_{fi} 'S AND THEN SQUARE THE SUM TO INSERT IN THE TRANSITION RATE FORMULA. ONE FINAL COMMENT HERE: THE FACTOR $1/(E_r - E_i + i\epsilon)$ IS JUST THE FOURIER TRANSFORM OF THE FREE PARTICLE PROPAGATOR $e^{i(E_r - E_i)t}$. SO BY TRANSFORMING INTO MOMENTUM SPACE OUR RULES FOR COMPUTING AMPLITUDES STAYS QUITE SIMPLE.

LIGHT SCATTERING FROM AN ATOM

NOW LET'S RETURN TO THE PROBLEM OF LIGHT SCATTERING FROM AN ATOM. IF THE INCIDENT RADIATION SCATTERS OFF THE ATOM WITHOUT EXCITING ATOM, HOW CAN THIS PROCESS BE DESCRIBED? ASSUME THE ATOM STARTS IN THE GROUND STATE AND ENDS UP IN THE GROUND STATE.

THE INCOMING PHOTON HAS A MOMENTUM AND POLARIZATION $\vec{k}_1, \vec{\epsilon}_1$ AND THE OUTGOING PHOTON HAS MOMENTUM AND POLARIZATION $\vec{k}_2, \vec{\epsilon}_2$. THE DIAGRAM ON THE RIGHT DESCRIBES THE PROCESS DESCRIBED.

THE INTERACTIONS OCCUR TWICE BUT ONLY 1 PHOTON IS EXCHANGED AT A TIME. FIRST A PHOTON IS ABSORBED AND THEN A SECOND EMITTED. SINCE WE KNOW THE FORMULA FOR SINGLE PHOTON ABSORPTION AND EMISSION, WE CAN WRITE DOWN M_{fi}



$$(H_{INT})_{fk} \rightarrow \left(\frac{e \vec{p}_0 \cdot \vec{e}_2^*}{mc \sqrt{2\omega_{k_2}}} e^{-i\vec{k}_2 \cdot \vec{x}_0} \right)_{on}$$

And

$$(H_{INT})_{ki} \rightarrow \left(\frac{e \vec{p}_0 \cdot \vec{e}_1}{mc \sqrt{2\omega_{k_1}}} e^{+i\vec{k}_1 \cdot \vec{x}_0} \right)_{no}$$

The INTERMEDIATE STATE OF THE SYSTEM, n , MUST BE SUMMED OVER TO INCLUDE ALL POSSIBLE INTERACTIONS. The MATRIX ELEMENTS $()_{on}$ AND $()_{no}$ DENOTE TRANSITIONS BETWEEN THE EXCITED STATE n AND THE ~~INITIAL~~ STATE 0 AND THE ~~INITIAL~~ STATE 0 TO EXCITED STATE n RESPECTIVELY. WE HAVE THE ACCORDING TO OUR RULE FOR DETERMINING M_{fi}

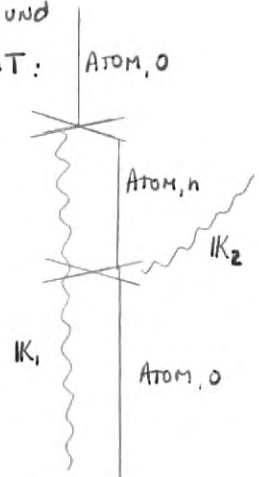
$$M_{fi}^{k_1, k_2} = \sum_n \left(\frac{e \vec{p}_0 \cdot \vec{e}_2^*}{mc \sqrt{2\omega_{k_2}}} e^{-i\vec{k}_2 \cdot \vec{x}_0} \right)_{on} \frac{1}{(E_n - E_0 - \hbar\omega_{k_1})} \left(\frac{e \vec{p}_0 \cdot \vec{e}_1}{mc \sqrt{2\omega_{k_1}}} e^{+i\vec{k}_1 \cdot \vec{x}_0} \right)_{no}$$

HAVE WE SUMMED OVER ALL POSSIBLE INTERMEDIATE STATES? NO!

THERE IS ANOTHER WAY THAT THE SAME PROCESS CAN OCCUR. THIS TIME THE PHOTON k_1 DOES NOT ACTUALLY INTERACT WITH THE ATOM BUT RATHER THE ATOM EMITS A PHOTON k_2 AND THEN TURNS AROUND AND ABSORBS IT BACK. THE DIAGRAM FOR THE PROCESS IS ON THE RIGHT:

THE AMPLITUDE FOR THIS PROCESS IS

$$M_{fi}^{k_1, k_2} = \sum_n \left(\frac{e \vec{p}_0 \cdot \vec{e}_1}{mc \sqrt{2\omega_{k_1}}} e^{+i\vec{k}_1 \cdot \vec{x}_0} \right)_{on} \frac{1}{(E_n + \hbar\omega_1 + \hbar\omega_2 - E_0 - \hbar\omega_1)} \left(\frac{e \vec{p}_0 \cdot \vec{e}_2^*}{c \sqrt{2\omega_{k_2}}} e^{-i\vec{k}_2 \cdot \vec{x}_0} \right)_{no}$$



THE DENOMINATOR TELLS US THAT INITIALLY

$$E_i = E_0 + \hbar\omega_1$$

I.E., THE ATOM IS IN ENERGY STATE E_0 AND THE PHOTON EXISTS.

AND THE INTERMEDIATE STATE

$$E_k = E_n + \hbar\omega_1 + \hbar\omega_2$$

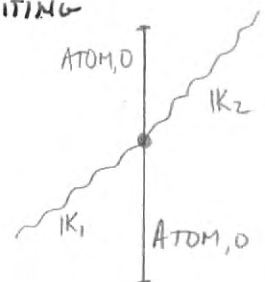
I.E., THE ATOM IS IN ENERGY STATE E_n AND BOTH PHOTONS EXIST. WE MUST ADD THE TWO AMPLITUDES TOGETHER TO GET THE TOTAL AMPLITUDE.

~~Done~~

NOW DO WE HAVE EVERYTHING COMPUTED RIGHT? NO!! WHAT ABOUT THE $\vec{A} \cdot \vec{A}$ TERM? WHAT DOES THIS TERM DO? IN THIS PROCESS PHOTON k_1 IS ABSORBED AND k_2 IS EMITTED WITHOUT EXCITING THE ATOM. THE AMPLITUDE FOR THIS TRANSITION IS

$$\frac{e^2}{mc^2} \vec{e}_2 \cdot \vec{e}_1 \left(e^{-i(\vec{k}_2 - \vec{k}_1) \cdot \vec{x}_0} \right)_{00}$$

THUS WE HAVE THREE TERMS TO ADD TOGETHER TO GET THE COMPLETE AND ACCURATE SCATTERING AMPLITUDES.



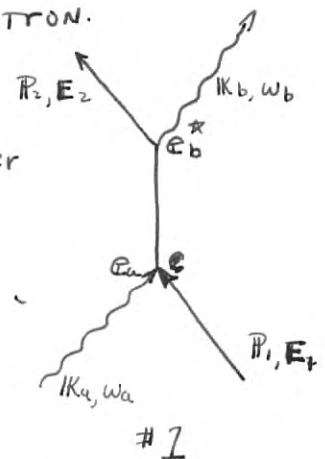
43. LIGHT SCATTERING FROM A RELATIVISTIC ELECTRON

I'd now like to work out the problem of light scattering from an atom where the incident photon has an energy comparable to the rest mass of an electron. For the energy range of interest the gamma rays have such a high momentum that the possibility exists to produce a pair, i.e. an electron and positron. We are interested here in the case of the energetic photon hitting an electron which is free (its energy is greater than the binding energy of BeV). So the problem is to see what happens when a sluggish electron gets clobbered by a powerful photon.

We shall assume the electron has momentum P_i and the photon has ^{REST} energy equal to $m_e c^2$ where m_e is the mass of the electron. The relativistic energy of the electron is then $E = \sqrt{P_i^2 + m_e^2}$ where $c=1$. The photon is described by a frequency, ω_a , and wave number, k_a . The photon is further described by a particular polarization vector ϵ_a . For each value of k_a there are two possible values for ϵ_a , each being at right angles to each other. The problem is to figure out what happens when the photon interacts with the electron.

The diagram for this interaction is to the right:

The diagram tells us that an electron of momentum P_i , energy E_i absorbs a photon of frequency ω_a , wave number k_a (i.e., it is scattered by the vector potential of the quantum field); then the electron propagates as a free particle with energy $E_i + \hbar\omega_a$ until finally the electron emits a photon of energy $\hbar\omega_b$, wave number k_b and polarization ϵ_b^* .



I want to clarify, before I go on, what I mean when I refer to a relativistic or non-relativistic problem. The non-relativistic problems relate to systems where all the velocities are low ~~vs~~ as compared to the speed of light. When I am

DEALING RELATIVISTIC PROBLEMS I WILL PRIMARILY BE WORKING IN RELATIONSHIPS WHICH CAN BEST BE DESCRIBED AS MANIFESTLY RELATIVISTICALLY INVARIANT. HERE I AM STEALING SCHWINGER'S TERMINOLOGY BECAUSE IT CORRECTLY ~~DEFINES~~ DEFINES THE SITUATION. BY THIS EXOTIC NOMENCLATURE WE REFER TO A METHOD OF ANALYSIS WHICH IN ITSELF IS NOT RELATIVISTICALLY INVARIANT, ONLY BECAUSE IT DOESN'T APPEAR TO BE; BUT ULTIMATELY THE SOLUTION, AND THE ANALYSIS, IS TRULY RELATIVISTICALLY INVARIANT. THIS IS A TREMENDOUS HELP IN WORK OUT PROBLEMS IN Q.E.D. BECAUSE RELATIVISTIC TERMINOLOGY GETS VERY CONFUSING AND HARD TO KEEP STRAIGHT UNLESS YOU KNOW WHAT YOU ARE DOING. I SHALL BE WORKING WITH THE MANIFESTLY RELATIVISTIC NOTATION AND SHOW YOU HOW TO DEMONSTRATE THAT THE ANSWER IS ACTUALLY INVARIANT.

WE CAN NOW START THE PROBLEM BY ASKING WITH WHAT PROBABILITY THE SYSTEM HAS TO START IN ITS INITIAL STATE i AND END IN THE FINAL STATE AS DESCRIBED BY THE DIAGRAM ON THE PREVIOUS PAGE. FROM OUR RULE FOR CALCULATING THE TRANSITION RATE,

$$\text{Prob. } \frac{\text{TRANSITION}}{\text{SEC}} = \frac{2\pi}{\hbar} |M_{fi}|^2 \delta(E_z - E_i - \omega_a + \omega_b)$$

THE DELTA FUNCTION CONTAINS THE CONSERVATION OF ENERGY CRITERIA FOR THE REACTION TO GO, IN ADDITION MOMENTUM MUST BE SATISFIED THEREFORE

$$p_2 + \hbar k_b = p_1 + \hbar k_a$$

WHERE I LOOSELY INTERCHANGE k_b AND ω_b .

TO CALCULATE THE TRANSITION MATRIX ELEMENT M_{fi} WE HAVE JUST LEARNED FROM OUR LAST LECTURE THAT

$$M_{fi} = \sum_r H'_{fr} \frac{1}{E_i - E_r + i\epsilon} H'_{ri}$$

HERE WE SUM OVER ALL POSSIBLE INTERMEDIATE STATES r WHERE THE ELECTRON HAS ENERGY $p_1 + \omega_a$. WE MUST CALCULATE THE INTERACTION TERMS FOR THE PROCESS. THE PROCESS, BY THE WAY, IS COMMONLY REFERRED TO AS THE COMPTON EFFECT. THIS EFFECT CONSISTS OF THE SCATTERING OF ENERGETIC PHOTONS BY A FREE ELECTRON.

SINCE WE ARE DEALING WITH THE RELATIVISTIC TREATMENT OF THE ELECTRON, WE MUST DERIVE THE INTERACTION TERM FROM THE DIRAC HAMILTONIAN. IN GENERAL THE DIRAC EQUATION IS GIVEN BY

$$\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = [\vec{\alpha} \cdot (-i\vec{\nabla} + \frac{e}{c} \vec{A}) + \beta m + e A_t] \psi$$

WHERE $A_t = V$ THE SCALAR POTENTIAL. THIS EQUATION DOESN'T LOOK INVARIANT IN FORM BUT IT IS GOOD FOR WORKING OUT QUANTUM MECHANICAL PROBLEMS. IT IS NOT IN AN OBVIOUSLY INVARIANT FORM. LET'S SEE WHAT CAN BE DONE TO MAKE IT LOOK INVARIANT. THE DIRAC HAMILTONIAN IS GIVEN BY

$$H_{D,A} = \vec{\alpha} \cdot (\vec{p} - \frac{e}{c} \vec{A}) + \beta m + A_t(x,t)$$

WHICH SOLVES THE ABOVE EQUATION, VIZ

$$\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = H_{D,A} \psi$$

IF WE MULTIPLY BOTH SIDES OF THIS EQUATION AND REARRANGE THE TERMS WE GET

$$[\beta (\frac{\hbar}{i} \frac{\partial}{\partial t} - e A_t) - \beta \vec{\alpha} \cdot (i\vec{\nabla} - \frac{e}{c} \vec{A})] \psi = \beta^2 m \psi = m \psi$$

WHERE $\beta^2 = \gamma_t^2 = 1$ AND $\beta \alpha_x = \gamma_x$, $\beta \alpha_y = \gamma_y$, $\beta \alpha_z = \gamma_z$. AND $\alpha_{x,y,z} = \gamma_t \gamma_{x,y,z}$. γ_μ IS A FOUR VECTOR WHICH WE HAVE DISCUSSED PREVIOUSLY AND WHOSE PROPERTIES AND COMMUTATION RULES ARE KNOWN. WE CAN SUBSEQUENTLY REDUCE THE DIRAC EQUATION TO THE RELATIVISTICALLY APPEARING FORM

$$[\gamma_\mu (p_\mu - \frac{e}{c} A_\mu)] \psi = m \psi$$

WE THEREFORE HAVE DEMONSTRATED THE INVARIANCE OF THE ORIGINAL EXPRESSION FOR THE DIRAC EQUATION.

RETURNING TO THE INTERACTION PIECE THAT WE WANT, IN THE ABSENCE OF AN EXTERNAL POTENTIAL THE DIRAC HAMILTONIAN, $H_{D,0}$, IS JUST

$$H_{D,0} = \vec{\alpha} \cdot \vec{p} + \beta m$$

SUBTRACTING $H_{D,0}$ FROM $H_{D,A}$ WE GET THE INTERACTION TERM

$$H' = -\frac{e}{c} \vec{\alpha} \cdot \vec{A}(x) + A_t(x,t)$$

FOR NOW WE WILL ONLY CONSIDER A VECTOR POTENTIAL $\vec{A}(x)$, I.E., $A(x)$ IMPLIES THAT THE INTERACTION HAMILTONIAN IS INFLUENCED BY THE QUANTUM FIELD VARIABLES WHILE A_t IMPLIES THE PRESENCE OF ATOMIC VARIABLES CREATING THE POTENTIAL.

Now the solution to the Dirac equation for a free electron in the absence of a potential is given by the solution

$$\psi = u e^{-i\vec{p}\cdot\vec{x}}$$

which produces

$$(\vec{\alpha}\cdot\vec{p} + \beta m) u = E_p u$$

When the plane wave piece $e^{-i\vec{p}\cdot\vec{x}}$ is taken out. The function u is four ~~ket~~ by four matrix describing the photon energy and momentum. We are interested in computing the matrix element between states 1 and 2 with state 3 being the intermediate state. This intermediate state, or virtual state, has momentum $P_3 = P_1 + k_a$. Now we can write the matrix element M_{fi}

$$M_{fi} = \sum_3 \frac{e^2}{c^2} \frac{\overbrace{(u_2^* (\vec{\alpha}\cdot\vec{e}_b^*) u_3)}^{H_{fk}} \cdot 1 \cdot \overbrace{(u_3^* (\vec{\alpha}\cdot\vec{e}_a) u_1)}^{H_{ki}}}{\sqrt{2\omega_b} (E_{p_1} + k_a - E_{p_3} + i\epsilon) \sqrt{2\omega_a}}$$

Reading right to left the matrix element says that electron in state u_1 absorbs the photon of polarization in direction \vec{e}_a ; then it propagates as a free electron of energy $E_{p_1} + k_a - E_{p_3}$ whereupon it emits a photon of polarization \vec{e}_b^* (the star denotes emission) and finally we ask to find the electron in one of its eigenstates u_2^* .

How do we calculate H_{ki} ? Well, we know what the vector potential is, $\vec{A} = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} [\vec{e}_{\vec{k}} a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \vec{e}_{-\vec{k}} a_{-\vec{k}} e^{i\vec{k}\cdot\vec{x}}] \hat{z}$

$$H_{ki} = \int \psi_R \left\{ \vec{\alpha}\cdot \sum_{\vec{k}} \frac{\vec{e}_{\vec{k}}}{\sqrt{2\omega_{\vec{k}}}} [\vec{e}_{\vec{k}} a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \vec{e}_{-\vec{k}} a_{-\vec{k}} e^{i\vec{k}\cdot\vec{x}}] \right\} d^3x$$

In the expression for the vector potential $a_{\vec{k}}$ and $a_{-\vec{k}}^*$ are, respectively, the operators for the emission and absorption by the k^{th} oscillator of a photon. But to evaluate the integral we must know what ψ_i and ψ_R are. Well ψ_i denote the composite wave function for the system where the electron is in state 1 and ^{all} the photons are in the field. ψ_R denotes the state where the electron is excited to state 3 and the photons are in the k^{th} mode.

$$\psi_i = \psi_1 \phi_0$$

$$\psi_f = \psi_3 \phi_k$$

$$H'_{ki} = \int \psi_3^* \phi_k^* H_{int} \phi_l \psi_1 d^3x$$

SUBSTITUTING IN FOR H_{int}

$$H'_{ki} = \int \psi_3^* \phi_k^* \left\{ \sum_L \frac{e}{c} \frac{\vec{\alpha} \cdot \vec{a}_L^* \vec{e}_L^*}{\sqrt{2\omega_L}} e^{iL \cdot x} + \frac{e}{c} \frac{\vec{\alpha} \cdot \vec{a}_L \vec{e}_L}{\sqrt{2\omega_L}} e^{-iL \cdot x} \right\} \phi_l \psi_1$$

SINCE a_L WORKS DIRECTLY ON ϕ_L TO KILL THE L^{th} PHOTON TO k_a

$$\int \phi_k^* a_L \phi_L d^3x = \sqrt{\frac{n+1}{2\omega_{ka}}} \quad n = \text{NUMBER OF PHOTONS WE HAVE}$$

IF WE ONLY HAVE ONE PHOTON PRESENT IN THE PROCESS

$$H'_{ki} = \sum_L \int \phi_k^* a_L \phi_L \int \psi_3^* (\vec{\alpha} \cdot \vec{e}_L) e^{-iL \cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

SINCE a_L^* HAS NO IMMEDIATE SIGNIFICANCE.

THUS

$$H'_{ki} = \frac{1}{\sqrt{2\omega_{ka}}} \int \psi_3^* (\vec{\alpha} \cdot \vec{e}_a) \psi_1 \frac{d^3x}{\sqrt{2\omega_L}}$$

WE NOW MUST SUBSTITUTE IN THE INTEGRAL THE FREE PARTICLE WAVE FUNCTIONS, I.E.,

$$H'_{ki} = \frac{1}{\sqrt{2\omega_{ka}}} \int U_3^* e^{-iP_3 \cdot x} (\vec{\alpha} \cdot \vec{e}_a) e^{+iK_a \cdot x} U_1 e^{iP_1 \cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

OR

$$= \frac{1}{\sqrt{2\omega_{ka}}} \int U_3^* (\vec{\alpha} \cdot \vec{e}_a) U_1 e^{-i[P_3 + K_a - P_1] \cdot x} \frac{d^3x}{\sqrt{2\omega_L}}$$

THE INTEGRATION OVER d^3x PRODUCES THE DELTA FUNCTION $\delta(P_3 - K_a - P_1)$ WHICH IS NOTHING MORE THAN A STATEMENT OF THE CONSERVATION OF MOMENTUM. IF YOU DIDN'T REMEMBER TO WRITE IT DOWN INITIALLY, THE RULE SAYS YOU DON'T GO ANYWHERE UNLESS YOU CONSERVE MOMENTUM. THE INTEGRATION OVER THE VOLUME HAS BEEN NORMALIZED TO $V=1$ FOR SIMPLICITY. THE SECOND PART OF THE MATRIX ELEMENT, I.E., THE H'_{fk} IS FOUND IN A SIMILAR WAY AS H'_{ki} . THIS TIME HOWEVER WE WILL USE THE a_L^* OPERATOR TO EMIT THE PHOTON AGAIN. THUS WE GET THE FIRST TERM ON THE RIGHT HAND SIDE OF M_{fi} , I.E.

$$H'_{fk} = \frac{1}{\sqrt{2\omega_{kb}}} U_2^* (\vec{\alpha} \cdot \vec{e}_b^*) U_3$$

THIS COMPLETES M_{fi} WHEN THE SUMMATION IS TAKEN OVER ALL THE VIRTUAL STATES 3.

IT IS NOT OBVIOUS FROM THE SUMMATION OVER ALL THE INTERMEDIATE STATES THAT WE FORGOT SOMETHING - BUT WE DID. THERE IS ANOTHER WAY THE SCATTERING CAN HAPPEN AND THAT PROCESS HAS THE ELECTRON EMITTING A PHOTON AND SUBSEQUENTLY ABSORBING THE INCIDENT PHOTON. THE DIAGRAM FOR THIS PROCESS IS TO THE RIGHT. THE MATRIX ELEMENT FOR THIS PROCESS CAN BE WRITTEN DOWN RIGHT AWAY FROM OUR KNOWLEDGE OF THE LAST EXERCISE, I.e.

$$M_{fi}^{(2)} = \sum_4 \frac{e^2}{c^2} \frac{[u_2^*(\alpha \cdot \epsilon_a) u_4] [u_4^*(\alpha \cdot \epsilon_b^*) u_1]}{\sqrt{2\omega_a} [E_{p_1} + \hbar\omega_a - (E_{p_4} + \hbar\omega_a + \hbar\omega_b)] \sqrt{2\omega_b}}$$

THE DENOMINATOR CAN BE PUT IN A DIFFERENT FORM SINCE ENERGY MUST BE CONSERVED, I.E. $E_{p_1} + \hbar\omega_a = E_{p_2} + \hbar\omega_b$ SO THAT

$$M_{fi}^{(2)} = \sum_4 \frac{e^2}{c^2} \frac{[u_2^*(\alpha \cdot \epsilon_a) u_4] [u_4^*(\alpha \cdot \epsilon_b^*) u_1]}{\sqrt{2\omega_a} [E_{p_2} - E_{p_4} - \hbar\omega_a] \sqrt{2\omega_b}}$$

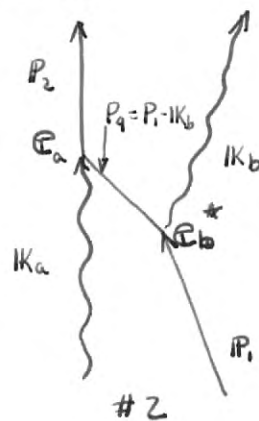
THE OTHER MATRIX ELEMENT FOR PROCESS 1 IS

$$M_{fi}^{(1)} = \sum_3 \frac{e^2}{c^2} \frac{[u_2^*(\alpha \cdot \epsilon_b^*) u_3] [u_3^*(\alpha \cdot \epsilon_a) u_1]}{\sqrt{2\omega_b} [E_{p_1} - E_{p_3} + \hbar\omega_a]}$$

THE TOTAL MATRIX ELEMENT FOR THE SCATTERING IS THE SUM OF THE TWO PIECES: $M_{fi}^{(1)} + M_{fi}^{(2)}$.

NOW SOMETHING USUAL OCCURS WHEN SUMMING OVER ALL POSSIBLE INTERMEDIATE STATES - THE DIRAC EQUATION ALLOWS FOR POSITIVE AND NEGATIVE ENERGY VALUES. WHAT THAT MEANS IS THERE IS A PROBABILITY TO MAKE A TRANSITION INTO A NEGATIVE ENERGY STATE. WHILE THIS SEEMS LIKE A PHYSICALLY STUPID IDEA, WE CAN PURSUE THE IDEA TO SEE WHAT HAPPENS. WHAT WE WILL FIND OUT IS THAT THE NEGATIVE ENERGY STATES CORRESPOND TO PAIR PRODUCTION.

LET'S BEGIN BY SUMMING OVER THE INTERMEDIATE STATE FOR PROCESS 1 AND BREAK THE SUMMATION INTO TWO PARTS: ONE OVER POSITIVE ENERGIES OR MOMENTA AND THE OTHER PART SUMS OVER NEGATIVE ENERGIES.



$$M_{fi}^{(1)} = \sum_{P_3 > 0} \frac{[U_2^*(\alpha \cdot e_b^*) U_3] [U_3^*(\alpha \cdot e_a) U_1]}{\sqrt{4\omega_a \omega_b} [(E_{p_1} + K_a) - E_{p_3}]} + \sum_{P_3 < 0} \frac{[U_2^*(\alpha \cdot e_b^*) U_3] [U_3^*(\alpha \cdot e_a) U_1]}{\sqrt{2\omega_a \omega_b} [(E_{p_1} + K_a) + E_{p_3}]}$$

WHERE $E_{p_3} = \pm \sqrt{m^2 c^4 + c^2 P_3^2}$

BY PUTTING THE TWO PIECES OVER A COMMON DENOMINATOR THE SUMMATION CAN BE TAKEN OVER ALL VIRTUAL STATES OF ENERGY E_{p_3} , I.E., OVER ALL STATES OF U_3 WHICH INCLUDES 4 (2 FOR EACH SPIN STATE AND 2 FOR EACH ENERGY STATE).

$$M_{fi}^{(1)} = \sum \frac{[U_2^*(\alpha \cdot e_b^*) U_3] (E_{p_1} + K_a + E_{p_3}) [U_3^*(\alpha \cdot e_a) U_1]}{\sqrt{4\omega_a \omega_b} [(E_{p_1} + K_a)^2 - E_{p_3}^2]} + \sum \frac{[U_2^*(\alpha \cdot e_b^*) U_3] (E_{p_1} + K_a - E_{p_3}) [U_3^*(\alpha \cdot e_a) U_1]}{\sqrt{4\omega_a \omega_b} [(E_{p_1} + K_a)^2 - E_{p_3}^2]}$$

THE DENOMINATOR NO LONGER DEPENDS ON THE SIGN OF THE ENERGY.

NOW WE USE THE FACT THAT

$$(\alpha \cdot P_3 + \beta m) U_3 = \pm E_{p_3} U_3$$

IS A SOLUTION TO THE DIRAC EQUATION TO PERMIT PUTTING BOTH PIECES OVER THE COMMON DENOMINATOR

$$M_{fi}^{(1)} = \sum_{\text{ALL } U_3} \frac{[U_2^*(\alpha \cdot e_b^*) (E_{p_1} + \omega_a + \alpha \cdot P_3 + \beta m) U_3] [U_3^*(\alpha \cdot e_a) U_1]}{\sqrt{4\omega_a \omega_b} [(E_{p_1} + K_a)^2 - E_{p_3}^2]}$$

SINCE U_3 FORMS A COMPLETE SET OF WAVE FUNCTIONS THE SUM OVER ALL U_3 MUST BE 1, I.E., $\sum_{U_3} U_3 U_3^* = 1$

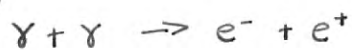
$$M_{fi}^{(1)} = U_2^*(\alpha \cdot e_b^*) \frac{(E_{p_1} + \hbar \omega_a + \alpha \cdot P_3 + \beta m)}{\sqrt{4\omega_a \omega_b} [(E_{p_1} + \omega_a)^2 - E_{p_3}^2]} (\alpha \cdot e_a) U_1$$

PAIR PRODUCTION THEORY

PAIR PRODUCTION, NEGATIVE SEA AND SPACE-TIME EXPLANATION

THERE IS ANOTHER WAY TO LOOK AT COMPTON SCATTERING WHICH IS A LITTLE MORE SUBTLE IN ITS PHYSICAL CONNOTATION BUT IS MATHEMATICALLY, AND PHYSICALLY, ANALOGOUS TO COMPTON SCATTERING; THAT IS THE IDEA OF PAIR PRODUCTION. SINCE PAIR PRODUCTION IS INTERESTING I WOULD LIKE TO GIVE IT A LITTLE MORE ATTENTION AT THIS TIME.

PAIR PRODUCTION CAN RESULT IN SEVERAL WAYS: A SINGLE PHOTON OF ENERGY GREATER THAN $2m$ CAN INTERACT WITH A NUCLEAR FIELD EITHER BY THE PHOTON CREATING A PAIR AND THE ELECTRON INTERACTING WITH THE FIELD OR BY THE PHOTON CREATING THE PAIR WITH THE POSITRON INTERACTING WITH THE FIELD. ALTERNATELY TWO PHOTONS CAN ANNIHILATE AND CREATE AN ELECTRON AND POSITRON. IN THE LATTER CASE ONE PHOTON CAN BE SUPPLIED BY THE COULOMB FIELD OF THE NUCLEUS, I.E.,



THE OPPOSITE REACTION HERE IS PAIR ANNIHILATION WHERE THE ELECTRON AND POSITRON PRODUCE TWO GAMMA RAYS.

THE INTERESTING THING ABOUT PAIR PRODUCTION WAS THAT THE EXISTENCE OF THE POSITRON WAS PREDICTED BY THE DIRAC EQUATION. IT IS A CONSEQUENCE OF THE DIRAC WAVE EQUATION THAT PARTICLES CAN HAVE ENERGIES GIVEN BY

$$E = \pm \sqrt{m^2 c^4 + p^2 c^2}$$

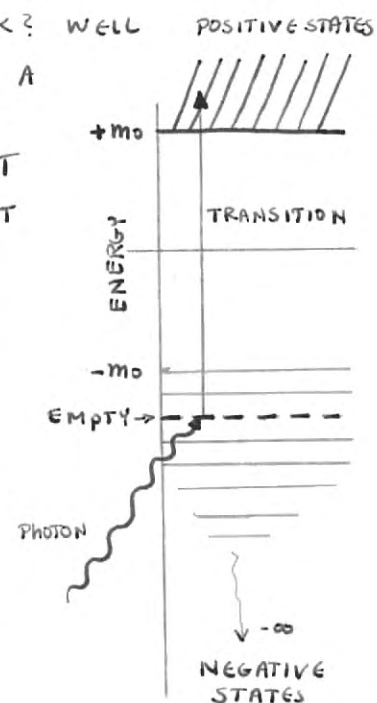
WHERE FOR A PHOTON OF ZERO REST MASS $\omega = k$. THE POSITIVE ENERGY DESCRIBES THE ELECTRON WITH ORDINARY CHARACTER WHILE THE NEGATIVE ENERGY SOLUTION DESCRIBES A NEW PARTICLE.

THE IDEA, FIRST PROPOSED BY DIRAC, TO EXPLAIN THE NEGATIVE ENERGY ROOT CONSIDER A NEGATIVE SEA OF STATES WHICH THE POSITIVE ELECTRON OR PARTICLE COULD MAKE TRANSITIONS INTO AND FROM. FOR AN ELECTRON TO GET KNOCKED OUT OF THE NEGATIVE SEA IT HAD TO RECEIVE $2m_0 c^2$ FROM THE ENERGETIC PHOTON. IN ESSENCE AN ELECTRON AND A HOLE ARE CREATED; THE HOLE BEING THE POSITRON. DIRAC IMPROPERLY IDENTIFIED THE HOLE AS

A PROTON. IN 1932 ANDERSON EXPERIMENTALLY DETECTED THE POSITRON.

ONE DIFFICULTY WITH THE NEGATIVE SEA IDEA IS THAT THE PROCESS NECESSARY TO PRODUCE THE PAIR IN ADDITION IS A SOURCE OF ENERGY. THIS MAKES IT SEEM LIKE AN IMPOSSIBLE THEORY. BUT THE DIRAC EQUATION WAS SO SUCCESSFUL IN PREDICTING THE ATOMIC SPECTRUMS INCLUDING SPIN ~~AT~~ IN A RELATIVISTIC CONDITION THAT THE THEORY MUST BE RIGHT. A WAY OUT OF THE DILEMMA IS TO PROPOSE THAT THE NEGATIVE STATES ARE NEARLY FULL. A VACUUM STATE CAN BE DEFINED AS ONE WHERE ALL THE NEGATIVE STATES ARE FULL (DON'T WORRY ABOUT THE INFINITE AMOUNT OF ENERGY NEEDED TO FILL THE BOTTOMLESS PIT) AND ALL THE POSITIVE STATES ARE EMPTY. WHAT STOPS TRANSITIONS BETWEEN POSITIVE AND NEGATIVE STATES IS THE EXCLUSION PRINCIPLE. AN ELECTRON, OF EITHER SIGN, CANNOT GO INTO ITS ANTI-STATE IF IT IS OCCUPIED.

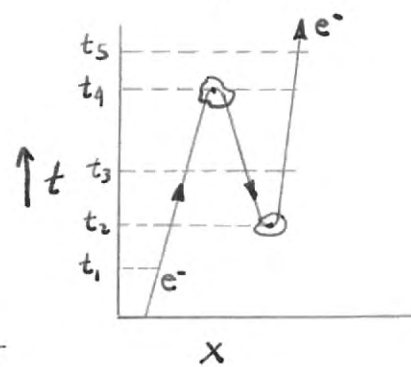
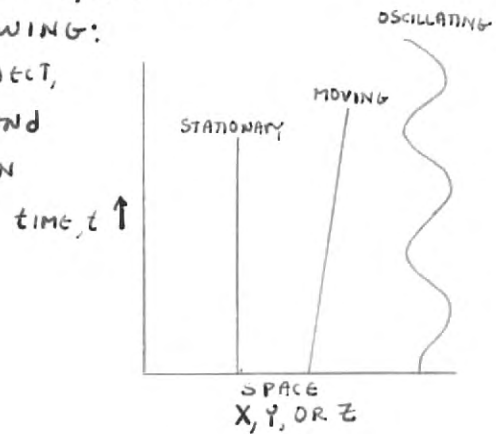
NOW DIRAC ARGUED THAT IF THE ELECTROMAGNETIC FIELD ~~WIT~~ HAD ENOUGH ENERGY IT COULD LIFT THE NEGATIVE ELECTRON, SOMETIMES CALLED THE NEGATON, INTO A POSITIVE ENERGY STATE. FOR THIS TO HAPPEN TWO PHOTONS ARE NEEDED TO CONSERVE ENERGY AND MOMENTUM. HOW DOES THIS PROCESS LOOK? WELL AN ELECTRON OF POSITIVE ENERGY IS CREATED IN ADDITION A NEGATIVE STATE IS LEFT EMPTY IN THE SEA. THE EMPTY STATE OR HOLE IS LIKE A WATER BUBBLE. IT CAN FLOAT AROUND CARRYING CHARGE. IT IS NECESSARY TO CANCEL OUT THIS CHARGED STATE BY SOME TRICK. DIRAC DIDN'T KNOW HOW TO FILL THE HOLE BACK UP BY THE DECAY OF AN ELECTRON INTO THE HOLE. IT WAS LEFT TO OPPENHEIMER TO DEMONSTRATE THE SYMMETRY OF THE TRANSITION AND THE IDEA OF PAIR ANNIHILATION WAS LATER VERIFIED EXPERIMENTALLY BY ANDERSON. UNFORTUNATELY THIS HOLE-BUSINESS IS PHYSICALLY DIFFICULT TO WORK WITH AND FOR SOME TIME, BETWEEN 1929 AND 1949, QED DIDN'T MAKE MUCH PROGRESS DUE TO THE DIFFICULTIES ENCOUNTERED.



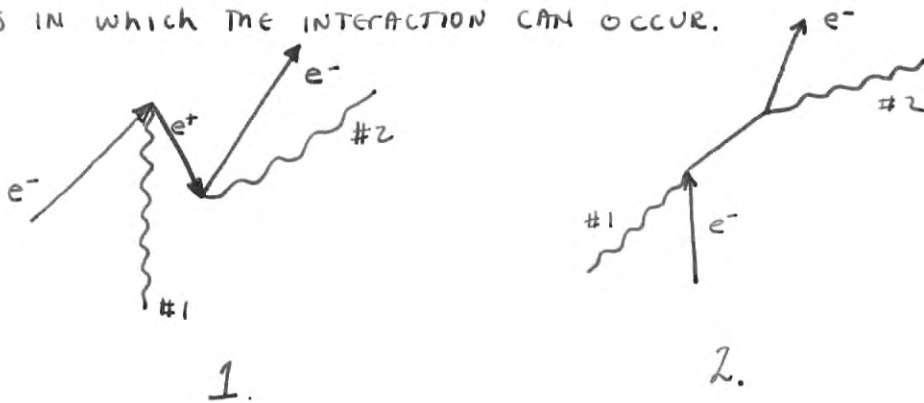
IN '48-'49 FEYNMAN PROPOSED HIS SPACE-TIME ARGUMENTS FOR WORKING OUT THESE QED PROBLEM. SINCE THEN THE MACHINERY OF SPACE-TIME PHYSICS HAS PROVEN TO BE A GREAT TOOL IN MANIPULATING AND SOLVING SUCH PROBLEMS AS PAIR PRODUCTION. THE SIMPLE CONCEPT OF A PARTICLE MOVING IN SPACE-TIME CAN BE REDUCED TO A 2-DIMENSIONAL DRAWING:

THE 3 MOTIONS SHOWN INDICATE A STATIONARY OBJECT, ONE THAT IS MOVING UNDER AN APPLIED FORCE, AND ANOTHER ONE WHICH IS OSCILLATORY DUE TO AN EXTERNAL FIELD. A PARTICLE FOLLOWING A SPACE TIME TRAJECTORY CAN BE FOUND TO BE MOVING FORWARD AS WELL AS BACKWARD IN TIME AS SEEN IN THE OTHER DIAGRAM:

HERE THE ELECTRON IS MOVING FORWARD IN TIME AT t_1 . BUT AT t_2 SUDDENLY TWO NEW PARTICLES APPEAR UNTIL AT t_4 SUDDENLY TWO PARTICLES CONVERGE, ANNIHILATE, AND THE ELECTRON ~~GOES~~ A NEW ELECTRON CONTINUES ON AT t_5 . THE PARTICLE MOVING BACKWARD IN TIME CORRESPONDS TO THE POSITRON. IN ORDER TO TURN THE ELECTRON AROUND IN TIME IT IS NECESSARY TO HAVE A FIELD PRESENT TO SCATTER OFF OF.



IN THE CASE OF COMPTON SCATTERING THERE ARE TWO WAYS IN WHICH THE INTERACTION CAN OCCUR.



44. MANIFESTLY INVARIANT THEORY OF PAIR PRODUCTION

I'd now like to go into the quantitative theory of pair production. In the process of we will come across the three major rules making up all of Q.E.D. In order to develop the theory in a relativistically simple form which is easy to compare with the old theory of scattering I'd like to introduce some new notation.

The 4-vector γ_μ has previously been defined so that it satisfies the commutation rule

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu}$$

If a_μ is another 4-vector satisfying this same rule, then define the matrix \not{a} to be

$$\not{a} = a_\mu \gamma_\mu = a_t \gamma_t - a_x \gamma_x - a_y \gamma_y - a_z \gamma_z$$

This matrix is relativistically invariant and therefore will be quite useful. In the same way we can define the momentum 4-vector p_μ in a relativistic form, i.e., $\not{p} = p_\mu \gamma_\mu$. Accordingly the Dirac equation can be expressed as,

$$i \gamma_\mu (-i p_\mu) u = p_\mu \gamma_\mu u = m u \quad \rightarrow \quad \not{p} u = m u$$

One final notation rule for two invariant matrices \not{a} and \not{b} the following is true:

$$\not{a} \not{b} = - \not{b} \not{a} + 2(a \cdot b) \quad \text{where } a \cdot b = a_\mu b_\mu$$

and

$$\not{a} \not{a} = a \cdot a$$

Now we write down one of our important propagation rules

RULE: AN ELECTRON PROPAGATES FROM ONE INTERACTION TO ANOTHER VIRTUALLY WITH AN AMPLITUDE PROPORTIONAL TO $\frac{1}{\not{p} - m + i\epsilon}$

THIS FREE PARTICLE PROPAGATOR WAS OBTAINED PREVIOUSLY BY TAKING THE FOURIER TRANSFORM OF THE POSITION PROPAGATOR.

THUS WE WILL BE WORKING IN THE MOMENTUM-ENERGY REPRESENTATION RATHER THAN IN SPACE-TIME; IT MAKES THINGS A LOT EASIER. SINCE THERE MAY BE SOME CONCEPTUAL DIFFICULTIES IN UNDERSTANDING THE OPERATION OF THE MATRIX \not{p} IN THE DENOMINATOR, THE FACTOR CAN BE REWRITTEN AS

$$\frac{1}{\not{p} - m + i\epsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$

NOW THE DENOMINATOR IS A PURELY SCALAR QUANTITY WITH A SMALL IMAGINARY PIECE WHICH WILL EVENTUALLY BE OMITTED. THE STRENGTH OF THE INTERACTION DEPENDS ON THE DEVIATION OF THE ENERGY OF THE PHOTON, \not{p} , AND ELECTRON, m , ENERGY.

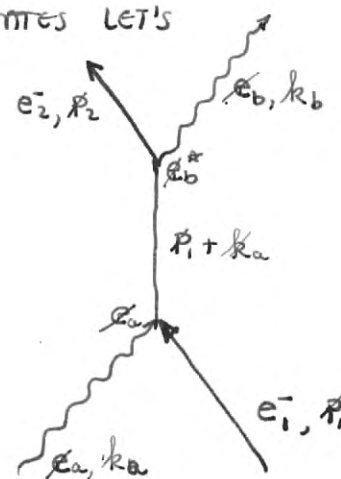
IF THE INCIDENT PHOTON IS TAKEN TO BE A VECTOR POTENTIAL WITH POLARIZATION ϵ_μ THEN THE COUPLING OF THE ELECTRON WITH THE PHOTON IS GIVEN BY $\not{\epsilon}$.

RULE: (i). THE AMPLITUDE TO ABSORB A PHOTON IS $\not{\epsilon}$
 (ii) THE AMPLITUDE TO EMIT A PHOTON IS $\not{\epsilon}^*$

WITH THE RULE FOR COUPLING THE ELECTRON WITH THE FIELD AND THE RULE FOR VIRTUAL PROPAGATION IN THE EXCITED STATES LET'S LOOK AGAIN AT THE COMPTON SCATTERING.

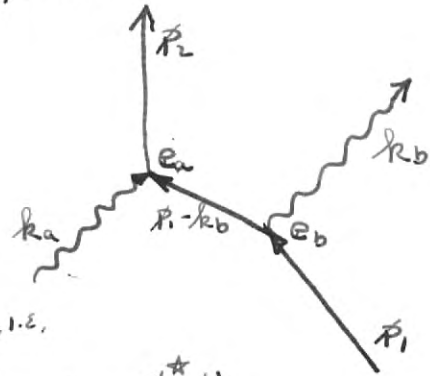
CONSIDER THE DIAGRAM TO THE RIGHT. IN THE MOMENTUM REPRESENTATION, THE MATRIX ELEMENT FOR THE PROCESS CAN BE COMPUTED DIRECTLY FROM THE SECOND ORDER PERTURBATION TERM (THIS IS REQUIRED BECAUSE THE ELECTRON INTERACTS TWICE). IF THE ELECTRON STARTS IN STATE U_1 , ABSORBS PHOTON k_a , PROPAGATES WITH ENERGY $\not{p} + k_a$, EMITS PHOTON k_b , AND PROPAGATES WITH MOMENTUM \not{p}_2 INTO STATE \bar{U}_2 , THE AMPLITUDE TO DO ALL THIS IS FOUND TO BE PROPORTIONAL TO

$$\bar{U}_2 \not{\epsilon}_b^* \frac{1}{\not{p} + k_a - m + i\epsilon} \not{\epsilon}_a U_1$$



TO THIS DIAGRAM WE MUST ADD THE AMPLITUDE TO EMIT PHOTON b FIRST SINCE THE TIMING OF THE EVENT IS NOT IMPORTANT. LATER WE WILL SEE THAT THIS IS JUST A STATEMENT OF THE RELATIVISTIC INVARIANCE OF THE PROCESS. THE AMPLITUDE FOR THIS SECOND PROCESS TO OCCUR IS

$$\bar{u}_2 \not{\epsilon}_a \frac{1}{\not{p}_1 - \not{k}_b - m + i\epsilon} \not{\epsilon}_b^* u_1$$



IN ORDER TO CONSERVE MOMENTUM AND ENERGY IN THE PROCESS $\not{p}_2 = \not{p}_1 - \not{k}_b + \not{k}_a$. THE TOTAL MATRIX ELEMENT IS THE SUM OF THESE TWO TERMS, I.E.

$$M_{12} = \bar{u}_2 \not{\epsilon}_b^* \frac{1}{\not{p}_1 + \not{k}_a - m + i\epsilon} \not{\epsilon}_a u_1 + \bar{u}_2 \not{\epsilon}_a \frac{1}{\not{p}_1 - \not{k}_b - m + i\epsilon} \not{\epsilon}_b^* u_1$$

HERE I HAVE SLIPPED IN THE 4-VECTOR $k (\omega, k_x, k_y, k_z)$ IN PREPARATION OF CONVERTING THIS RESULT TO A MANIFESTLY INVARIANT FORM. RIGHT NOW IT IS A REAL MESS; IT'S A MIXTURE OF OUR OLD TECHNIQUE OF COMPUTING MATRIX ELEMENTS TOGETHER WITH SOME MYSTERIOUS 4-VECTOR NOTATION. SOMEHOW THIS ALL ENDS UP TO BE INVARIANT AND I'D NOW LIKE TO SEE IF I CAN PROVE IT IS.

NOW I FORGOT SOME FACTORS IN WRITING THE ABOVE MATRIX ELEMENT WHICH I NOW MUST STRAIGHTEN OUT. FIRST THE STRENGTH OF THE COUPLING OF THE ELECTRON IS PROPORTIONAL TO $\sqrt{\frac{4\pi e^2}{2\omega_a}}$ FOR THE ABSORPTION OF THE PHOTON AND PROPORTIONAL TO

$\sqrt{\frac{4\pi e^2}{2\omega_b}}$. Thus M_{12} IS PROPORTIONAL TO $\sqrt{\frac{4\pi e^2}{2\omega_a} \cdot \frac{4\pi e^2}{2\omega_b}}$. LET ME DEFINE THE POLARIZATION 4-VECTOR ϵ_μ AS

$$\epsilon_\mu \gamma_\mu = \not{\epsilon} \sqrt{4\pi e^2}$$

Therefore,

$\not{\epsilon}_a \not{\epsilon}_b^* 4\pi e^2$ IS THE COUPLING STRENGTH OF THE PHOTON AND ELECTRON. WE STILL HAVE TO WORRY ABOUT THE FACTOR $(\frac{1}{2\omega_a} \cdot \frac{1}{2\omega_b})^{1/2}$ BECAUSE IT IS NOT IN AN INVARIANT FORM. SO LET'S REDEFINE M_{12} .

Let's define an invariant matrix element M^R which is related to the old matrix element M computed by the non-relativistic perturbation theory. The two M 's are related by a scale factor $(\frac{1}{2\omega_a 2\omega_b})^{1/2}$, i.e.,

$$M^{\text{RELATIVISTIC}} = M^{\text{old Theory}} \sqrt{2\omega_a 2\omega_b}$$

We have another mistake because the wave functions u_i were assumed to obey the normalization criteria $u_i^* u_i = 1$. But in the relativistic case the plane wave density is no longer invariant as the box shrinks. That is to say as the density of the box increases due to shrinking box dimensions the energy content starts to increase proportionally as the speed increases. Thus we have an unrelativistic choice of normalization constants which can be corrected for by choosing the normalization to be

$$u_i^* u_i = 2E_i$$

In this equation we have set the 4th component of the 4-vector current equal to the 4th component of the momentum 4-vector P_μ . From our earlier work $\bar{u} = u^* \beta$ where $\beta = \gamma_t$ and $\bar{u} \beta = u^* \beta^2 = u^*$. Thus $u_i^* u_i = \bar{u} \beta u_i = \bar{u} \gamma_t u = 2E$. The space components of the current four vector are normalized to $u_i^* u_i = 2m$ such that

$$\bar{u} \gamma_x u = 2P_x \quad \bar{u} \gamma_y u = 2P_y \quad \bar{u} \gamma_z u = 2P_z$$

We now have a relativistically invariant normalization of the wave functions. We do this by scaling all the u 's up by a factor $\sqrt{E/E_0}$ which make the matrix elements bigger proportionately. To take this scale change into account we must divide it out. Thus while this factor is not in itself invariant it is just the right size to make the answer come out relativistically invariant.

FROM THE OLD SCATTERING THEORY (I.E. THE NONRELATIVISTIC CASE) THE TRANSITION RATE COULD BE EXPRESSED IN TERMS OF THE SCATTERING CROSS SECTION AS

$$\frac{\text{TRANSITION RATE}}{\text{SEC}} = \sigma v_{\text{RELATIVE}} = \left[2\pi \delta(E_2 + \omega_b - \omega_a - E_1) (2\pi)^3 \delta^3(\vec{p}_2 + \vec{k}_b - \vec{k}_a - \vec{p}_1) \right] \cdot \left[|M_{fi}|^2 \frac{d^3 k_b}{2\omega_b (2\pi)^3} \cdot \frac{d^3 p_2}{2E_2 (2\pi)^3} \right] \cdot \left[\frac{1}{2E_1 \cdot 2\omega_a} \right]$$

I HAVE ADDED A FEW EXTRA TERMS IN THE EXPRESSION PREVIOUSLY USED. THE DELTA FUNCTION IS NOW A 4-DIMENSIONAL ONE IN TERMS OF THE 4-MOMENTUM; IT DESCRIBES THE CONSERVATION OF ENERGY AND MOMENTUM. I DID THIS TO KEEP THE ANSWER MANIFESTLY INVARIANT. THE INTEGRATION OVER $d^3 k_b$ IS OVER THE FINAL MOMENTUM STATES OF THE PHOTON. THE $d^3 p_2$ TERM WAS PUT IN SO THE 4-DIMENSIONAL DELTA FUNCTION IS COMPLETE. NOW THE RESULT HAS A CHANCE OF BEING INVARIANT. THE VELOCITY IS THE RELATIVE VELOCITY BETWEEN THE PARTICLES. IN WORDS THE TRANSITION RATE IS

$$\text{RATE} = \text{CROSSSECTION} \times \text{RELATIVE VELOCITY} = \left\{ \begin{array}{l} 2\pi \times 4\text{-DIMENSIONAL DELTA FUNCTION} \times \\ \text{MATRIX ELEMENT SQUARED DIVIDED BY} \\ \text{TWICE THE ENERGY OF EVERY PARTICLE IN FOOT} \end{array} \right.$$

$$\text{rate} = (2\pi)^4 \delta^4(p_{2\mu} + k_{b\mu} - p_{1\mu} - k_{a\mu}) \times (M_{fi})^2 \times \frac{1}{2E_1 \cdot 2\omega_a} \times \frac{d^3 k_a}{2\omega_a (2\pi)^3} \frac{d^3 p_2}{2E_2 (2\pi)^3}$$

SINCE THE FORM OF THIS EQUATION CONTAINS THE RELATIVE VELOCITY OF THE PARTICLES INVOLVED IN THE COLLISION IT DOES NOT APPEAR TO BE RELATIVISTICALLY INVARIANT. BUT WE CAN DEMONSTRATE THE MANIFESTLY INVARIANT FORM OF THE RATE EQUATION BY SHOWING THAT

$$v_{\text{rel}} (2E_1, 2\omega_a) = (v_1 - v_2) (2E_1, 2\omega_a) = \left[-\vec{p}_1^2 / E_1^2 + (\vec{p}_1 \cdot \vec{k}_a)^2 / E_1^2 \omega_a^2 \right]^{1/2}$$

IS AN INVARIANT EXPRESSION. TO PROVE THE ABOVE EQUALITY I'LL WORK IN ONE DIMENSION TO SAVE SOME WORK. THUS THE 4-VECTOR \vec{p}_1 HAS THE COMPONENTS $\vec{p}_1 = E_1, p_{1x}$ BUT $p_{1x} = mV_1 = E_1 V_1$; THEREFORE $\vec{p}_1 = E_1, E_1 V_1$. SIMILARLY THE 4-VECTOR \vec{k}_a HAS THE COMPONENTS $\omega_a, \omega_a v_a$. HERE I INCLUDE v_a FOR COMPLETENESS; FOR A PHOTON $v_a = c = 1$.

INSERTING \vec{p}_i AND k_a IN TO THE EQUATION

$$-\vec{p}_i^2 k_a^2 + (\vec{p}_i \cdot \vec{k}_a)^2 = -(E_i^2 - E_i^2 v_i^2) (\omega_a^2 - \omega_a^2 v_a^2) + (E_i \omega_a - E_i \omega_a \vec{v}_i \cdot \vec{v}_a)^2$$

EXPANDING AND CONSIDERING $\vec{v}_i \cdot \vec{v}_a = 0$, I.E., THE PARTICLES COLLIDE HEADON WE CAN COLLECT TERMS AND HAVE LEFT

$$-\vec{p}_i^2 k_a^2 + (\vec{p}_i \cdot \vec{k}_a)^2 = [E_i \omega_a (v_a - v_i)]^2$$

THE SQUARE ROOT THEN GIVES JUST THE RIGHT ANSWER WITH THE FACTOR OF 2 ADDED TO SCALE THE ENERGIES RIGHT; THUS

$$4E_i \omega_a (v_a - v_i) = [-\vec{p}_i^2 k_a^2 + (\vec{p}_i \cdot \vec{k}_a)^2]^{1/2}$$

I'D NOW LIKE TO CONSIDER THE FACTOR $d^3 k_a d^3 p_f$ AND SHOW IT CAN BE EXPRESSED IN THE MANIFESTLY INVARIANT FORM OF $2\pi \delta(k_a^2 - 0) \frac{d^4 k_a}{(2\pi)^4} \int (p_f^2 - m^2) \frac{d^4 p_f}{(2\pi)^4}$. CONSIDER NOW A SINGLE PARTICLE INTERACTION TO MAKE THINGS SIMPLER. WHEN WE SUBSTITUTE THIS INTO THE RATE FORMULA WE HAVE THE COMPLETE MANIFESTLY INVARIANT FORMULA WE DESIRE!

$$\sigma_{rel} = 2\pi \frac{1}{2E_i 2\omega_a} |M_{fi}^R|^2 \times (2\pi)^4 \delta^4(p_{2\mu} + k_{b\mu} - p_{1\mu} - k_{a\mu}) \times 2\pi \delta(p_i^2 - m^2) \frac{d^4 p_i}{(2\pi)^4}$$

IN GENERAL TERMINOLOGY

$$\text{TRANSITION RATE} = 2\pi \left[\pi \left(\frac{1}{2 \cdot \text{INCOMING PARTICLE ENERGY}} \right) \right] (\text{RELATIVISTIC MATRIX ELEMENT})^2 (2\pi)^4 \delta^4 \left(\sum_{\text{ALL IN}} p_{i\mu} - \sum_{\text{ALL OUT}} p_{o\mu} \right) \cdot \left[\pi \left(\frac{1}{2 \cdot \text{OUTGOING PARTICLE ENERGY}} \right) \right] \delta(p_{out}^2 - m^2) \frac{d^4 p_{out}}{(2\pi)^4}$$

IN SUMMARY WE HAVE CONSIDERED ALMOST ALL OF THE RULES OF QED IN DERIVING THIS MANIFESTLY INVARIANT SCATTERING THEORY. I'D LIKE TO DISCUSS NOW THE ONE RULE WE HAVEN'T TALKED ABOUT.

45. RULES OF QUANTUM ELECTRODYNAMICS

SO FAR WE HAVE DISCUSSED TWO OF THE THREE RULES OF Q.E.D.

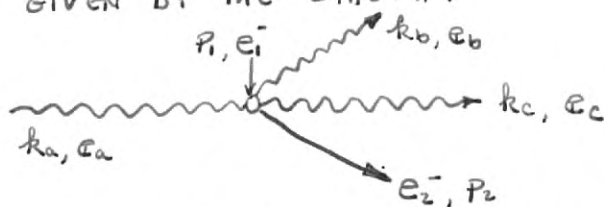
- RULES.
1. THE AMPLITUDE FOR A VIRTUAL e^- TO PROPAGATE IS $\propto \frac{1}{\not{p} - m + i\epsilon}$
 2. THE INTERACTION OF THE PHOTON AND ELECTRON IS COUPLED BY THE TERM $\sqrt{4\pi e^2} \not{A}$
 3. THE AMPLITUDE FOR A VIRTUAL PHOTON TO PROPAGATE IS PROPORTIONAL TO $\delta_{\mu\nu} / q^2 \delta_{\mu\nu} 4\pi e^2$

IT IS THE THIRD RULE WHICH WE HAVE NOT DISCUSSED AND I'D LIKE TO DO THAT NOW. THEN WE WILL HAVE ALL THE CONCEPTUAL TOOLS NEEDED TO WORK OUT ALL KINDS OF PROBLEMS. I WILL EMPHASIZE AS WE GO ON THE USE OF SPACE-TIME (FEYNMAN) DIAGRAMS TO WORK OUT THE VARIOUS PROBLEMS. THE DIAGRAMS SERVE AS A VERY USEFUL TOOL BECAUSE ONCE YOU ARE FAMILIAR WITH THEM AND CAN WRITE DOWN ALL THE \neq TOPOLOGICALLY DIFFERENT DIAGRAMS FOR A GIVEN PROBLEM, THEN IT IS NO LONGER NECESSARY TO WORK OUT THE MATHEMATICS; I.E., WRITE OUT ALL THE MATRIX ELEMENTS. BY STUDYING THE VARIOUS DIAGRAMS IT WILL BE POSSIBLE TO SEE WHICH DIAGRAM IS "STRONGER OR WEAKER" THAN THE OTHERS AND THUS APPROPRIATELY SIMPLY THE SITUATION. IF IT BECOMES NECESSARY TO COMPUTE THE EXPECTATION VALUES FOR THE PROCESS IN QUESTION, THEN THE DIAGRAM CAN BE CONVERTED DIRECTLY INTO THE NECESSARY FORM BY CAREFUL LABELING OF THE DIAGRAM.

~ TWO PHOTON EMISSION ~

AS AN EXAMPLE OF WHAT I MEAN, BEFORE I DEVELOPE RULE 3, LET'S CONSIDER A PROCESS WHICH MAY NOT HAVE BEEN OBSERVED.

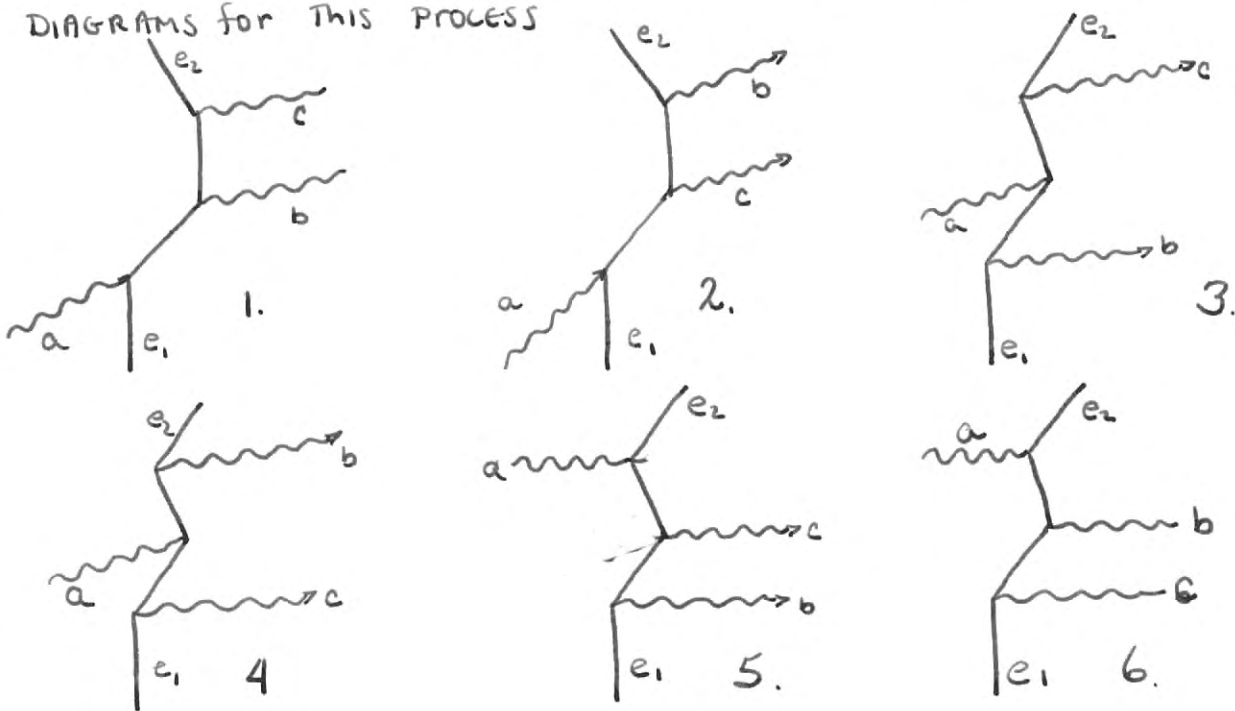
ASSUME WE HAVE A PHOTON SCATTERING OFF AN ELECTRON; WHAT IS THE PROBABILITY THAT TWO PHOTONS ARE SCATTERED OUT. THE PROCESS IS GIVEN BY THE DIAGRAM



LET'S ASSUME THE ELECTRON IS INITIALLY AT REST, I.E. $\mathbf{p}_1 = (m, 0, 0, 0)$.

IN ORDER TO CONSERVE MOMENTUM WE MUST HAVE THAT $\mathbf{p}_1 + \mathbf{k}_a = \mathbf{p}_2 + \mathbf{k}_b + \mathbf{k}_c$.

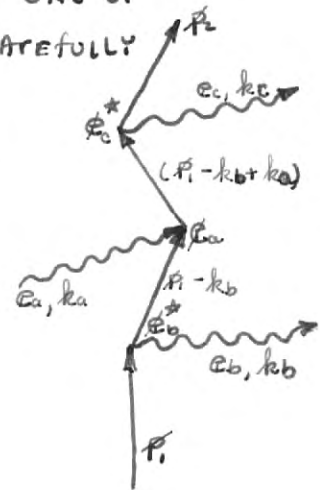
NOW WE DRAW ALL THE TOPOLOGICALLY DIFFERENT FEYNMAN DIAGRAMS FOR THIS PROCESS



WE HAVE THEN 6 DIAGRAMS WHICH ALL MUST BE ADDED TOGETHER. I SHOULD POINT THAT EACH OF THESE DIAGRAMS CAN BE DECOMPOSED INTO EIGHT MORE DIAGRAMS IF WE REVERSE THE TIME OCCURRENCE OF EACH EVENT. BUT SINCE OUR THEORY IS RELATIVISTICALLY INVARIANT WE DON'T NEED TO DO THAT. FOR EXERCISE LET'S WRITE ONE OF THESE DIAGRAMS OUT, SAY #3. LET'S DRAW IT AGAIN CAREFULLY LABELING EVERYTHING.

FIRST $\vec{p}_2 = \vec{p}_1 + \vec{k}_a - \vec{k}_b - \vec{k}_c$ OR $\vec{p}_2 + \vec{k}_c = \vec{p}_1 - \vec{k}_a + \vec{k}_b$
 WRITING OUT M_3

$$M_3 \propto \bar{u}_2 \epsilon_c^* \frac{1}{\vec{p}_1 - \vec{k}_b + \vec{k}_a - m\vec{1}} \epsilon_a \frac{1}{\vec{p}_1 - \vec{k}_b - m\vec{1}} \epsilon_b^* u_1$$

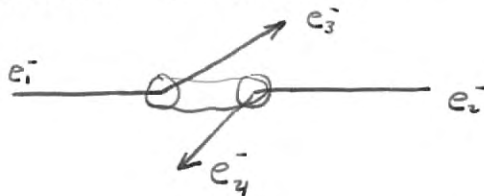


READING FROM RIGHT TO LEFT (THE WAY WE WRITE THE MATRIX ELEMENT OUT), THE ELECTRON OF MOMENTUM \vec{p}_1 IN STATE u_1 A. EMITS PHOTON k_b , PROPAGATES WITH ENERGY $\vec{p}_1 - \vec{k}_b$, ABSORBS PHOTON k_a , PROPAGATES WITH MOMENTUM $\vec{p}_1 - \vec{k}_b + \vec{k}_a$, FINALLY IT EMITS PHOTON k_c AND PROPAGATES ON WITH MOMENTUM $\vec{p}_2 + \vec{k}_c$ AND WE ASK THE PROBABILITY TO BE IN STATE \bar{u}_2 . SIMILARLY FOR THE OTHER 5 PROCESSES M 'S ARE COMPUTED SUCH THAT THE TOTAL RATE IS

$$\text{RATE} = \sigma V = 2\pi \left(\frac{1}{\underbrace{2E_1}_{\text{IN}} \underbrace{2\omega_a 2\omega_b 2\omega_c}_{\text{OUT}} 2E_2} \right) \cdot \left| \sum_{n=1}^5 M_n \right|^2 \times \text{DENSITY OF STATE}$$

ELECTRON-ELECTRON SCATTERING, THE RULE FOR VIRTUAL PHOTON PROPAGATION.

IN ORDER TO DEVELOPE THE RULE FOR VIRTUAL PHOTON PROPAGATION I WILL RETURN TO THE OLD WAY OF COMPUTING MATRIX ELEMENTS SO YOU CAN SEE BETTER WHERE IT COMES FROM. THE PROBLEM WE WANT TO CONSIDER IS TWO ELECTRON SCATTERING. IN THE REAL WORLD THE INTERACTION PICTURE LOOKS LIKE

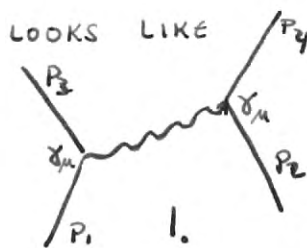


THE SCATTERING POTENTIAL IS DUE TO THE COULOMB INTERACTION OF THE TWO CHARGED PARTICLES. WE WILL CONSIDER THE CASE WHERE THE ELECTRONS MOVE FAST ENOUGH THAT THEIR MOTION IS AFFECTED BY RETARDATION EFFECTS. THUS WE MUST WORRY ABOUT Q.E.D WHICH ALLOWS FOR PHOTON EMISSION DURING THE EMISSION. OTHERWISE FOR SLOW MOVING ELECTRONS WE WOULD SUPPRESS THE RETARDATION EFFECTS AND CONSIDER THE STATIC INTERACTION OF THE COULOMB FIELD.

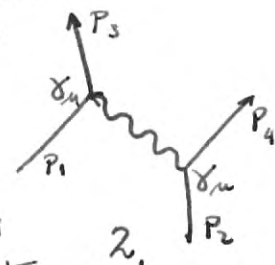
IN THE SPACE-TIME PICTURE THE INTERACTION LOOKS LIKE THE FOLLOWING:

HERE: THE ELECTRON 1 INTERACTS WITH THE COULOMB FIELD, RELEASING A PHOTON WHICH IS SUBSEQUENTLY ABSORBED BY THE OTHER INCOMING ELECTRON 2.

IN THIS PROCESS THE VACUUM STATE IS EXCITED BY THE VIRTUAL PHOTON BUT THE VACUUM STATE IS RESTORED AS THE PHOTON IS ABSORBED. IT SHOULD BE CLEAR THAT THIS PROCESS COULD HAPPEN ANOTHER WAY IF THE SECOND ELECTRON EMITS THE PHOTON FIRST AND THEN ELECTRON 1 ABSORBS IT.



THE δ_μ 'S WILL BE EXPLAINED IN A MINUTE. FOR PURPOSES OF SIMPLICITY WE WILL CONSIDER THE ELECTRONS AS DISTINGUISHABLE. WE COULD THINK OF ONE AS A MU MESON WHICH IS AN ELECTRON WITH A DIFFERENT MASS. THUS WE WON'T WORRY ABOUT THE EXCHANGE OF PARTICLES.

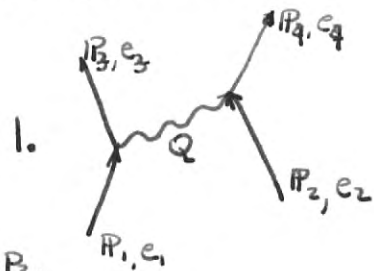


SINCE THIS IS A TWO INTERACTION PROCESS WE MUST HAVE THE INTERACTION TERM ACTING TWICE WHICH TELLS US WE WANT TO COMPUTE THE SECOND ORDER PERTURBATION MATRIX ELEMENT; I.E.,

$$M_{fi} = 4\pi e^2 \sum_k H_{fk} \frac{1}{E_i - E_k} H_{ki}$$

NOW WE ARE GOING TO WORK WITH THE OLD NON-RELATIVISTIC THEORY AND THEN SHOW IT IS MANIFESTLY INVARIANT. THE DESCRIPTION OF THE VARIOUS STATES OF THE SYSTEM IS AS FOLLOWS:

i, INITIAL	e_1^-, e_2^- ,	NO PHOTON
k, INTERMEDIATE	e_3^-, e_2^- ,	1 PHOTON
f, FINAL	e_3^-, e_4^- ,	NO PHOTON



THE INTERMEDIATE MOMENTUM OF THE PHOTON IS $Q = p_1 - p_3$.

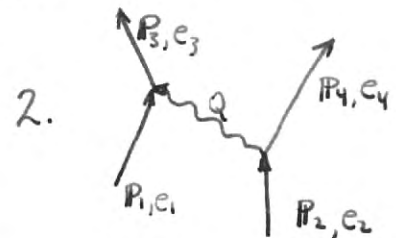
WE CAN WRITE DOWN THE MATRIX ELEMENT FOR THIS INTERACTION USING OUR RULES

$$M_{fi}^1 = \frac{4\pi e^2}{\sqrt{2\omega_Q} \sqrt{2\omega_Q}} (\bar{u}_4^* \alpha \cdot e^* u_2) \frac{1}{E_1 + E_2 - (E_3 + E_2 + \omega_Q)} (\bar{u}_3^* \alpha \cdot e u_1)$$

HERE $\omega_Q = |Q|$ SINCE $c=1$.

TO THIS TERM WE MUST ADD THE SECOND POSSIBLE AMPLITUDE,

$$M_{fi}^2 = \frac{4\pi e^2}{\sqrt{2\omega_Q} \sqrt{2\omega_Q}} (u_3^* \alpha \cdot e u_1) \frac{1}{E_1 + E_2 - (E_1 + E_4 + \omega_Q)} (u_4^* \alpha \cdot e^* u_2)$$



TO PUT THESE TERMS TOGETHER WE MUST PUT THEM OVER THE SAME DENOMINATOR. THIS CAN BE DONE USING THE CONSERVATION OF ENERGY RULE, I.E.,

$$E_1 + E_2 = E_3 + E_4 \quad \text{OR} \quad E_1 - E_3 = E_4 - E_2$$

FROM M_{fi}^1

$$\frac{1}{E_1 + E_2 - (E_3 + E_2 + \omega_Q)} = \frac{1}{E_1 - E_3 + \omega_Q}$$

FROM M_{fi}^2

$$\frac{1}{E_1 + E_2 - (E_1 + E_4 + \omega_Q)} = \frac{1}{E_2 - E_4 - \omega_Q} = \frac{1}{-(E_1 - E_3) - \omega_Q}$$

THE COMMON DENOMINATOR HERE IS $(E_1 - E_3)^2 - \omega_Q^2$.

$$M = M' + M'' = \frac{4\pi e^2}{2\omega q} \frac{[(E_1 - E_3) + \omega q] - [(E_1 - E_3) - \omega q]}{[(E_1 - E_3)^2 - \omega q^2]} (u_4^* (\alpha \cdot \epsilon^*) u_2) (u_3^* (\alpha \cdot \epsilon) u_1)$$

THIS CAN BE CONSIDERABLY SIMPLIFIED BY SUBTRACTING AND DIVIDING OUT THE NUMERATOR AND DENOMINATOR $2\omega q$ AND LEAVING

$$M = \frac{4\pi e^2}{(E_1 - E_3)^2 - \omega q^2} (u_4^* (\alpha \cdot \epsilon^*) u_2) (u_3^* (\alpha \cdot \epsilon) u_1)$$

NOW RECALL THAT $\omega q = |Q|$ SO $\omega q^2 = Q \cdot Q = (P_1 - P_3)^2$
 IF WE NOW IMAGINE THE PHOTON IS DESCRIBED BY A
 4-MOMENTUM WITH $q_4 = E_1 - E_3$ AND SPACE PART $\vec{Q} = P_1 - P_3$
 THEN $q_4^2 - \vec{Q} \cdot \vec{Q} = q^2$ SO THAT

$$M = \frac{4\pi e^2}{q^2} (u_4^* (\alpha \cdot \epsilon^*) u_2) (u_3^* (\alpha \cdot \epsilon) u_1)$$

THE STRENGTH OF THE INTERACTION DEPENDS ON THE DEVIATION OF q^2 FROM 0 MOMENTUM.

OUR ANSWER IS NOT COMPLETE IN THE ABOVE FORM BECAUSE WE MUST SUM OVER ALL POSSIBLE POLARIZATIONS ϵ_μ OF THE PHOTON

$$M = \sum_{\text{ALL POL}} \frac{4\pi e^2}{q^2} (u_4^* (\alpha \cdot \epsilon^*) u_2) (u_3^* (\alpha \cdot \epsilon) u_1)$$

THIS IS EQUIVALENT TO SUM OF THE FOUR POLARIZATIONS,

$$\frac{1}{q^2} (u_4^* \vec{\alpha}_{x,y,z} u_2) \cdot (u_3^* \vec{\alpha}_{x,y,z} u_1) + \frac{1}{q^2} (u_4^* \frac{\vec{\alpha} \cdot \vec{Q}}{Q} u_2) (u_3^* \frac{\alpha \cdot Q}{Q} u_1)$$

THIS DOESN'T LOOK RIGHT SINCE IT DOES NOT EQUAL THE ABOVE MATRIX ELEMENT; THERE IS AN EXTRA PIECE IN THE EXPRESSION INVOLVING $1/Q^2$. LET'S SEE IF WE CAN UNDERSTAND WHERE THIS EXTRA TERM CAME FROM BY FORCING THE EQUATION TO THE RELATIVISTICALLY INVARIANT. FIRST WE NEED THE FACT THAT $\bar{u}_\beta = u^*$ THEN WE HAVE

$$-\frac{1}{q^2} (\bar{u}_4 \beta \vec{\alpha} u_2) \cdot (\bar{u}_3 \beta \vec{\alpha} u_1) + \frac{1}{q^2} (u_4^* \frac{\vec{\alpha} \cdot \vec{Q}}{Q} u_2) (u_3^* \frac{\alpha \cdot Q}{Q} u_1)$$

Now $\beta \vec{\alpha} = \vec{\gamma}$ AND BY ADDING AND SUBTRACTING
 $(\bar{u}_4 \gamma_t u_2) + (\bar{u}_3 \gamma_t u_1)$ WE GET THAT

$$M = \frac{4\pi e^2}{q^2} (\bar{u}_4 \gamma_\mu u_2) (\bar{u}_3 \gamma_\mu u_1) + \frac{1}{q^2} \underbrace{(u_4^\dagger \frac{\alpha \cdot Q}{Q^2} u_2) (u_3^* \frac{\alpha \cdot Q}{Q^2} u_1) - (\bar{u}_4 \gamma_t u_2) (\bar{u}_3 \gamma_t u_1)}_{\text{debris}}$$

WE WOULD HAVE A NICE RELATIVISTICALLY INVARIANT RESULT IF
 SOMEHOW THE DEBRIS EQUALS ZERO. LET'S WORK ON THE
 TERM $u_3^* \frac{\alpha \cdot Q}{Q^2} u_1$; THIS EQUALS

$$u_3^* \frac{\alpha \cdot Q}{Q^2} u_1 = u_3^* (\alpha \cdot P_1 - \alpha \cdot P_3) u_1$$

BUT THE DIRAC EQUATION PREDICTS THAT $\alpha \cdot P_1 u_1 = E_1 u_1 - \beta m u_1$
 AND $\alpha \cdot P_3 u_3^* = E_3 u_3^* - \beta m u_3^*$ SO

$$u_3^* (\alpha \cdot P_1 - \alpha \cdot P_3) u_1 = u_3^* [(E_1 - \beta m) - (E_3 - \beta m)] u_1 \\ = u_3^* u_1 (E_1 - E_3)$$

SIMILARLY WE CAN SHOW THAT

$$u_4^\dagger \alpha \cdot Q u_2 = u_4^\dagger u_2 (E_4 - E_2)$$

SINCE $E_1 - E_3 = E_4 - E_2$

$$u_4^\dagger u_2 (E_4 - E_2) = u_4^\dagger u_2 (E_1 - E_3)$$

FINALLY THEN

$$(u_4^\dagger \frac{\alpha \cdot Q}{Q^2} u_2) (u_3^* \frac{\alpha \cdot Q}{Q^2} u_1) = u_4^\dagger u_2 u_3^* u_1 \frac{(E_1 - E_3)^2}{Q^2}$$

NOW THE TERM $(\bar{u}_4 \gamma_t u_2) (\bar{u}_3 \gamma_t u_1)$ CAN BE REWRITTEN SINCE
 $\beta = \gamma_t$ AND $\bar{u} \beta = u^\dagger$, I.E.,

$$(\bar{u}_4 \gamma_t u_2) (\bar{u}_3 \gamma_t u_1) = (u_4^\dagger u_2) (u_3^* u_1)$$

WE HAVE THEN

$$\text{debris} = u_4^\dagger u_2 u_3^* u_1 \frac{(E_1 - E_3)^2}{Q^2} - u_4^\dagger u_2 u_3^* u_1$$

$$= \frac{4\pi e^2}{q^2} \frac{1}{Q^2} u_4^\dagger u_2 u_3^* u_1 [(E_1 - E_3)^2 - Q^2]$$

RECALL THAT $q^2 = (E_1 - E_3)^2 - Q^2$ SO WE HAVE THAT

$$\text{DEBRIS} = \frac{U_2^* U_2 U_3^* U_1}{Q^2}$$

AND THIS MATRIX ELEMENT IS NON ZERO. DAMN! TOO BAD. WE MUST FIND AN EXPLANATION OF WHY IT APPEARS IN OUR FORMULA.

IF WE THINK BACK TO WHAT WE ASSUMED WHEN DEVELOPING THE RULE FOR COMPUTING THE INTERACTION AREA POTENTIAL WE EXPRESSED THE VECTOR POTENTIAL IN TERMS OF PLANE TRANSVERSE WAVES. WHEN WE DID THAT WE ASSUMED THE COULOMB GAUGE WHICH MEANS WE HAVE A SECOND TERM, THE STATIC, INSTANTANEOUS COULOMB INTERACTION BETWEEN THE PARTICLES. THE DEBRIS WHICH WE CALCULATED IS THE FOURIER TRANSFORM OF THIS STATIC INTERACTION TERM, e^2/r_{ij} , IN MOMENTUM SPACE, I.E., $\int \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{R} d^3R = \frac{4\pi}{Q^2}$

THIS TERM CORRESPONDS TO INSTANTANEOUS PROPAGATION OF THE VIRTUAL PHOTON,

WE HAVE THEN A TOTAL OF THREE DIAGRAMS WHICH MUST BE SUMMED TOGETHER TO OBTAIN THE FINAL RESULT. THE SUMMATION OVER THE FOUR POLARIZATIONS IS SEEN TO BE EQUIVALENT TO

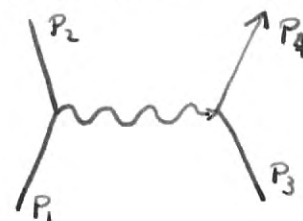
TRANSVERSE WAVES PLUS AN INSTANTANEOUS COULOMB INTERACTION.

HOWEVER SINCE OUR ANSWER IS INVARIANT UNDER A LORENTZ

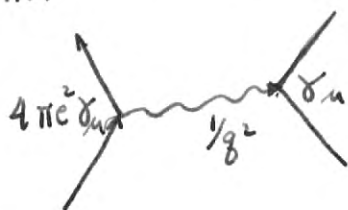
TRANSFORMATION, IT IS MEANINGFUL TO REDUCE THE FORMULA TO

THE TWO DIFFERENT DYNAMICAL INTERACTIONS INVOLVING THE

TRANSVERSE FIELD. THE DIAGRAM WE JUST COMPUTED IS, THEREFORE,



IT IS IGNORED AND WE REPLACE IT BY THE GENERAL PHOTON COUPLING DIAGRAM



46. HIGHER ORDER CORRECTION TERMS

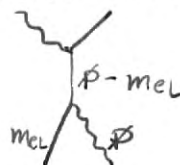
WE HAVE DEVELOPED A SET OF RULES FOR COMPUTING THE MATRIX ELEMENTS BETWEEN SOME INITIAL AND FINAL STATES. THE ANALYSIS OF THE INTERACTION HAS BEEN CONSIDERED IN A SEQUENCE OF PERTURBATION EACH WITH A STRENGTH OF $e^2/\hbar c \approx 1/137$. THE STUDY LED TO THE DEVELOPMENT OF THE RULE FOR COMPUTING THE SCATTERING RATES, I.E.,

$$\text{RATES} = \left(\prod_{\text{IN}} \frac{1}{2E_R} \right) |M_{fi}|^2 (2\pi)^4 \delta^4(\sum P_{\text{IN}} - \sum P_{\text{OUT}}) \cdot \prod_{\text{OUT}} \frac{d^4 P_{\text{OUT}}}{(2\pi)^4} \delta(P_{\text{OUT}}^2 - m_{\text{OUT}}^2)$$

THE RULES FOR CALCULATING M_{fi} WERE DERIVED AND WERE GIVEN IN THE FORM OF PROPAGATOR

VIRTUAL ELECTRON PROPAGATOR

$$\frac{1}{\not{p} - m_{eL}}$$

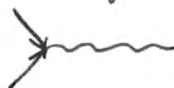


VIRTUAL PHOTON PROPAGATOR

$$4\pi e^2 \gamma_\mu \frac{1}{q^2} \gamma_\mu$$



REAL PHOTON EMISSION, $\frac{1}{4\pi e^2}$



THE RULE FOR SUMMING OVER ALL MOMENTA REQUIRES INTEGRATION OVER $d^4 k / (2\pi)^4$

FIRST ORDER PERTURBATION

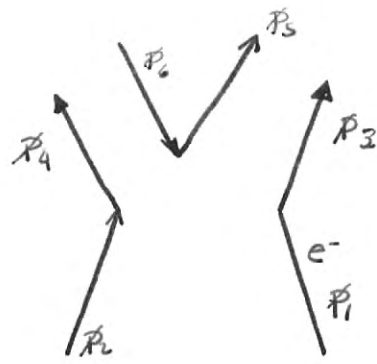
IF THE DESIRED ACCURACY OF A PROCESS IS ABOUT 1%, THEN THE FIRST ORDER PERTURBATION THEORY IS ADEQUATE, I.E., $(e^2/\hbar c)^1 = 1/137 \approx 1\%$. THIS LOWEST ORDER PERTURBATION IMPLIES THE INTERACTION ACTS ONLY ONCE. THEREFORE OUR DIAGRAMS CAN ONLY BE COUPLED BY DIAGRAMS ON THE RIGHT:

AS AN EXAMPLE OF HOW THE RULES WORK LET'S CONSIDER THE POSSIBILITY OF PAIR PRODUCTION FROM ELECTRON SCATTERING BY A NUCLEAR FIELD.

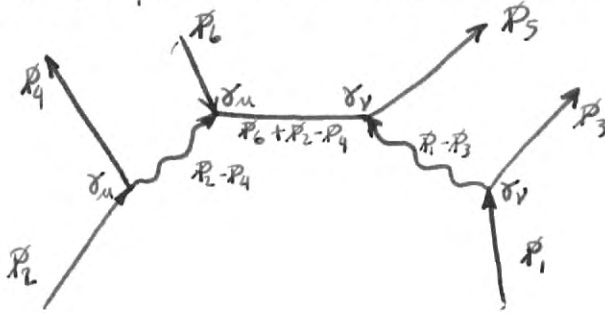


THE NUCLEAR FIELD IS ESSENTIALLY A STATIONARY ELECTRON FIELD AS CONSTRUCTED IN THE LAB. THUS THE PATHS IN THE CLOUD CHAMBER WOULD HAVE A FAST ELECTRON COMING IN WITH A PAIR BEING PRODUCED. THE ESSENTIALLY LINES OF THE DIAGRAM ARE DRAWN NEXT.

PAIR PRODUCTION FROM ELECTRON SCATTERING FROM A NUCLEAR FIELD.



THE PROBLEM IS HOW TO INTERCONNECT THE KINKS IN THE DIAGRAMS BECAUSE FOR EVERY KINK WE NEED A "WIRE" TO HOOK ONTO. SINCE OUR INTEREST IS ONLY TO FIRST ORDER, WE ONLY HAVE TO WORRY ABOUT SINGLE LINE INTERCONNECTS. SINCE THERE ARE 3 KINKS, THERE SHOULD BE 6 DIAGRAMS. ONE SUCH DIAGRAM IS



IN THIS DIAGRAM p_1 AND p_2 ARE THE PHYSICALLY INCOMING PARTICLES, WHILE $p_1 = (m_e, 0, 0, 0)$ AND p_4, p_5 ARE THE OUT GOING ELECTRONS. p_5 IS THE NEWLY CREATED ELECTRON WHILE p_6 IS IT'S CREATED PAIR AND REPRESENTS A PHYSICALLY OUTGOING POSITRON. MATHEMATICALLY p_6 IS CONSIDERED AN INCOMING PARTICLE. THE PAIR PRODUCTION WING OF THE DIAGRAM IS COUPLED TO THE COLLISION BY THE POLARIZED PHOTONS OF A VECTOR PROPAGATOR γ_u AND γ_v . WE CAN NOT WRITE DOWN M FOR THE PROCESS. ABOVE EACH TERM IN THE EXPRESSION FOR M I HAVE EXPRESSED FOR THE PART OF THE DIAGRAM WHICH IS WRITTEN OUT MATHEMATICALLY,

$$M = (\sqrt{4\pi e^2})^4 \left(\bar{u}_5 \gamma_\nu \cdot i \cdot \gamma_u u_6 \right) \frac{1}{p_6 + p_5 - p_4 - m_e c} \left(\bar{u}_3 \gamma_\nu u_1 \right) \left(\bar{u}_4 \gamma_u u_2 \right) \frac{1}{(p_2 - p_4)^2} \frac{1}{(p_1 - p_3)^2}$$

VIRTUAL ELECTRON PROPAGATION

V PHOTON EMISSION

M PHOTON EMISSION

VIRTUAL μ PHOTON PROPAGATION
 $\frac{1}{q^2} = \frac{1}{(p_2 - p_4)^2}$

VIRTUAL ν PHOTON PROPAGATION
 $\frac{1}{q^2} = \frac{1}{(p_1 - p_3)^2}$

4 for each photon exchange

IN WRITING THIS SCATTERING MATRIX ELEMENT WE MUST DECIDE WHAT TO DO WITH u_b , I.E., WHAT DO WE USE FOR IT. IF WE CALL THE MOMENTUM OF A POSITRON, e^+ , \vec{p} , THEN \vec{p}_b EQUALS THE NEGATIVE OF \vec{p} . SINCE $\vec{p}u = -m_{el}u$, $\vec{p}_b u_b = +m_{el}u_b$. CLEARLY WE ARE GETTING INTO DIFFICULTLY DISTINGUISHING BETWEEN PARTICLES WHICH ARE INCOMING AND OUTGOING BOTH PHYSICALLY AND MATHEMATICALLY.

LET'S DEFINE THE INITIAL AND FINAL STATE IN A TIME SENSE SUCH THAT THE INITIAL STATE OCCURS BEFORE THE FINAL STATE. ENTRANCE AND EXIT STATES WILL BE DEFINED IN A MATHEMATICAL SENSE. THE DIRECTION ARROWS WILL REFER TO ENTRANCE AND EXIT STATES. THUS THE INITIAL POSITRON IS IN THE EXIT STATE WHILE THE FINAL POS. ELECTRON IS IN THE ENTRANCE STATE. WITH THIS DEFINITION WE CAN EXPRESS M_{fi} AS THE PRODUCT OF $M_{EXIT} M_{ENTRANCE}$.

I SHOULD POINT OUT IN THE PROCESS JUST DESCRIBED IF THE DENOMINATORS ARE SMALL, I.E., $\vec{p}_2 \approx \vec{p}_4$ AND $\vec{p}_1 \approx \vec{p}_3$ THEN THE SCATTERING IS STRONG AND QED IS A GOOD PREDICTOR OF THE COUPLING SUCH THAT THE RATE CAN BE ACCURACY, PRECISELY, DETERMINED. THIS IS NOT VERY INTERESTING QED UNLESS YOU LIKE PRECISION. ON THE OTHER HAND WHEN $\vec{p}_2 - \vec{p}_4$ AND $\vec{p}_1 - \vec{p}_3$ ARE LARGE, IMPLYING VERY SHORT DISTANCE COUPLING, THEN QED GETS INTERESTING AND THE ACCURACY STARTS TO FALL APART. ERRORS ON THE ORDER OF 5 TO 10 PERCENT ARE MORE COMMON THAN ERRORS OF A FEW PART IN A MILLION.

SELF-ENERGY OF THE ELECTRON

I'D LIKE TO DISCUSS HIGHER ORDER CORRECTION TO THE FIRST ORDER PERTURBATION THEORY JUST DESCRIBED. IN PARTICULAR I'D LIKE TO DISCUSS THE COMPLICATION OF M_{fi} DUE TO THE DIAGONAL ELEMENTS OF H . IT IS THE NON-DIAGONAL ELEMENTS OF H , THE H_{fi} , ELEMENTS THAT WE HAVE USED TO COMPUTE THE TRANSITION RATES. THE H_{ii} IS NOW TO BE THE INTERACTION OF THE STATE WITH ITSELF. AS A RESULT M_{ii} IS NOT A RATE OF TRANSITION BUT RATHER A SHIFT IN ENERGY. BUT WE MUST BE CAREFUL BECAUSE THE NORMALIZATION IS OFF.

Suppose I WANT TO CALCULATE THE ENERGY SHIFT OF THE ELECTRON DUE TO ITS ACTION BACK ON ITSELF. I WANT THEN TO COMPUTE $\Delta E = U^* H_{ii} U$. WE KNOW THAT $U^* U = 2E$ SO LET'S DIVIDE THROUGH BY $U^* U$ AND GET

$$\frac{\Delta E}{2E} = \frac{U^* H_{ii} U}{U^* U}$$

OR $2E \Delta E = M_{ii}$

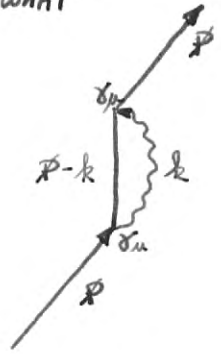
SINCE $E^2 = P^2 + m^2$ AND P IS THE SAME BEFORE AND AFTER THE PERTURBATION, THE CHANGE IN ENERGY IS EQUIVALENT TO A CHANGE IN MASS. WE CAN INTERPRET THE DIAGONAL ELEMENTS, M_{ii} , AS A CORRECTION TO THE MASS, ACTUALLY IT IS THE CORRECTION TO THE MASS SQUARED.

THE PHYSICAL INTERPRETATION OF THE SELF-ENERGY CORRECTION IS THAT AN ELECTRON AT REST HAS A FINITE PROBABILITY OF MAKING A VIRTUAL TRANSITION TO SOME NEW MOMENTUM STATE \not{p} PLUS SIMULTANEOUSLY EXISTING WITH A PHOTON. ULTIMATELY THE PHOTON IS REABSORBED BACK. SINCE THE PERTURBATION ~~IS~~ ACTS TWICE, THE CORRECTION IS SECOND ORDER, THE DIAGRAM FOR THE PROCESS IS TO THE RIGHT; UNFORTUNATELY WHEN THE CORRECTION IS COMPUTED, IT COMES OUT INFINITE. WAIT, THE ONE ELECTRON COULD HAVE BEEN AT REST AND THAT IS AN UNINVARIANT CALCULATION SO LET THE ELECTRON BE MOVING. NOW WE CORRECT FOR THE ENERGY DIFFERENCE BUT WE FIND THAT THE LORENTZ TRANSFORMATION DOES NOT GIVE THE RIGHT CORRECTION. WE'RE STILL IN TROUBLE. WHAT ABOUT THE COULOMB FIELD? DID WE FORGET IT? WELL e^2/r_{ii} IS LIKEWISE INFINITE SO NOW WE HAVE 2 INFINITE ENERGY RESULTS. HOW CAN WE PUT THEM TOGETHER SO JUST SO THEY BALANCE AND GIVE A RELATIVISTICALLY INVARIANT RESULT? ONE WAY, WHICH WE HAVE ALREADY DISCUSSED, IS TO CUT OFF THE FREQUENCY AT SOME ARBITRARY LARGE VALUE. BUT THIS APPROACH IS NOT MANIFESTLY INVARIANT. THIS INFINITE ENERGY PROBLEM PLAGUED QED UP TO 1949, AND GIVEN TODAY IT STILL BROTHERS US. BUT WE KEEP THE SHELL GAME GOING BY A LOT OF TRICKS TO GET AROUND THE PROBLEM BUT IT IS STILL WITH US.



LET'S TRY TO CALCULATE THE ENERGY SHIFT AND SEE WHAT HAPPENS, THE INTEGRAL OVER ALL PHOTONS IS

$$\Delta m^2 = M_{ii} = \int (\sqrt{4\pi e^2})^2 \frac{1}{k^2} \frac{\bar{u} \gamma_\mu \cdot 1 \cdot \gamma_\mu u}{(\not{P}-\not{k})-m+i\epsilon} \frac{d^4 k}{(2\pi)^4}$$



IT SHOULD BE EVIDENT HOW M_{ii} IS EXPRESSED. FROM PREVIOUS WORK. SINCE M_{ii} GETS SQUARED, I HAVE LEFT OUT THE i 'S WHICH GO ALONG; THEY ARE JUST UNESSENTIAL COMPLICATIONS.

NOW WHAT VALUE OF k DO WE USE, I.E., WHAT MODE IS EXCITED? SINCE IN A CAVITY RESONATOR ANY MODE MAY BE EXCITED, WE MUST INTEGRATE OVER ALL POSSIBLE MOMENTA OF THE VIRTUAL PHOTON. I SHOULD POINT OUT WE ARE WORRIED ONLY ABOUT FIRST ORDER CORRECTIONS HERE EVEN THOUGH A TWO PHOTON EMISSION AND REABSORPTION IS POSSIBLE.

HOW DO WE DO THE INTEGRAL? IT LOOKS FRIGHTENING SINCE THERE ARE MATRICES IN THE DENOMINATOR. LET'S PUT THEM IN THE NUMERATOR AND COLLECT THE IMAGINARY PIECE ϵ IN THE DEFINITION OF THE MASS, I.E.,

$$\bar{u} \gamma_\mu \frac{1}{\not{P}-\not{k}-m} \gamma_\mu u = \bar{u} \gamma_\mu \frac{(\not{P}-\not{k}+m)}{(\not{P}-\not{k})^2-m^2} \gamma_\mu u$$

WE CAN WORK ON THE RIGHT SIDE SOME BY OBSERVING IF THE γ_μ 'S WERE NEXT TO EACH OTHER THINGS WOULD BE COMPUTATIONALLY EASIER. THAT IS WE KNOW $\gamma_\mu \gamma_\mu = \gamma_t \gamma_t - \gamma_x \gamma_x - \gamma_y \gamma_y - \gamma_z \gamma_z = 1 - (-1) - (-1) - (-1)$ OR $\gamma_\mu \gamma_\mu = 4$. CLEARLY $\gamma_\mu m \gamma_\mu = m \gamma_\mu \gamma_\mu = 4m$. BUT WHAT ABOUT $\gamma_\mu (\not{P}-\not{k}) \gamma_\mu$. WE NEED THE RULE FOR $\gamma_\mu \not{a} \gamma_\mu$ WHICH YOU RECALL IS COMMUTATIVE BY THE 4 VECTOR COMMUTATION RULE.

$$\not{a} \not{b} = -\not{b} \not{a} + 2\mathbf{a} \cdot \mathbf{b}$$

THUS $\gamma_\mu \not{a} \gamma_\mu = \gamma_\mu (-\gamma_\mu \not{a}) + \gamma_\mu (2\mathbf{a}) = -4\mathbf{a} + 2\mathbf{a} = -2\mathbf{a}$
SO THAT

$$\bar{u} \gamma_\mu \frac{(\not{P}-\not{k}+m)}{(\not{P}-\not{k})^2-m^2} \gamma_\mu u = \bar{u} \frac{[-2\not{P}-2\not{k}+4m]}{(\not{P}-\not{k})^2-m^2} u$$

NOW $\not{P}u = mu$ SO WE HAVE $\bar{u} \frac{(2m-2\not{k})}{(\not{P}-\not{k})^2-m^2} u$

SINCE u IS NORMALIZED SUCH THAT $\bar{u}u = 2m$ WE HAVE THAT

$$+2m u \bar{u} = +2m^2$$

ALSO WE CAN SIMPLY $\bar{u} (\not{k}) u$ AS $2(\not{p} \cdot \not{k}) = 4\bar{p} \cdot \bar{k}$
 SINCE THE PHOTON HAS $m_\gamma = 0$. THE INTEGRAL OF INTEREST IS NOW

$$M = 4\pi e^2 \int \frac{4m^2 + 4\bar{p} \cdot \bar{k}}{(\not{p} - \not{k})^2 - m^2} \frac{d^4 k}{k^2 (2\pi)^4}$$

THE DENOMINATOR CAN BE SIMPLIFIED BY NOTING THAT $\not{p}u = mu$
 OR $\not{p}^2 = m^2$ THEREFORE

$$M = 4\pi e^2 \int \frac{4m^2 + 4\bar{p} \cdot \bar{k}}{(k^2 - 2\bar{p} \cdot \bar{k} + i\epsilon)} \frac{d^4 k}{k^2 (2\pi)^4}$$

IF WE ASSUME THE ELECTRON IS AT REST THEN $\bar{p} = (m, 0, 0, 0)$
 AND ASSUME $\bar{k} = (V, 0, 0, 0)$ SUCH THAT

$$M = 4\pi e^2 \int \frac{4m^2 + 4mV}{V^2 - 2mV + i\epsilon} \frac{dV d^3 k (2\pi)^{-4}}{k^2}$$

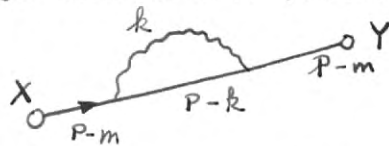
EVALUATING THIS INTEGRAL GIVES AN INFINITE VALUE.

47. MASS AND CHARGE RENORMALIZATION

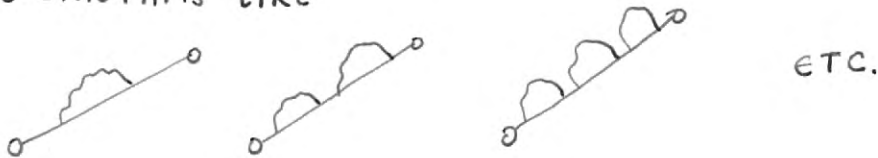
LAST TIME WE DISCUSSED THE INFINITE SELF-ENERGY OF THE ELECTRON BY WORKING OUT THE DIAGONAL ELEMENTS OF THE SCATTERING MATRIX M . THERE IS ANOTHER WAY TO CALCULATE THE SELF ENERGY USING THE DIAGRAM TECHNIQUE. AS AN EXAMPLE CONSIDER AGAIN THE COMPTON EFFECT AS THE BASIC REACTION AND NOW LET'S ASK HOW WE CAN CORRECT THE RESULTS BY GOING TO HIGHER ORDER IN THE PERTURBATION EXPANSION. THE SECOND ORDER CORRECTION LIMITS US TO ONE ADDITIONAL ELECTRON WHICH MUST BE EMITTED AND ABSORBED. THE POSSIBLE MODIFICATIONS TO THE FIRST ORDER PROCESS ARE MANY. IF THE VIRTUAL ELECTRON IS ALMOST FREE, I.E., THE PROPAGATOR IS CLOSE TO THE RIGHT MASS THEN THE INTERMEDIATE STATE CAN LAST A LONG TIME. IN THE INTERMEDIATE STATE MANY VIRTUAL PHOTONS CAN EXIST.



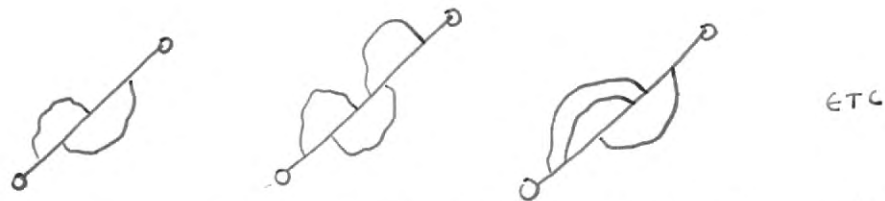
THE BASIC SECOND ORDER PROCESS IS THE FOLLOWING



HERE THE ELECTRON STARTS IN SOME STATE X , AND WE REALLY AREN'T CONCERNED WHAT THE INITIAL STATE IS, AND PROPAGATES TO SOME FINAL STATE Y . THUS $X \rightarrow Y$ COULD BE ANY ONE OF THE THREE STRAIGHT LINES IN THE COMPTON SCATTERING DIAGRAM. IF THE ELECTRON IS NEARLY FREE, I.E., $P^2 = m^2$, THEN THERE MAY BE MORE PHOTONS EXISTING. IF WE CONCERN OURSELVES, FIRST, WITH ONLY ONE VIRTUAL PHOTON EXISTING AT A TIME, WE HAVE POSSIBLE DIAGRAMS LIKE



LESS LIKELY, BUT POSSIBLY OCCURRING, TWO AND MORE VIRTUAL PHOTONS CAN EXIST SIMULTANEOUSLY. THEN WE GET DIAGRAMS LIKE,



EVEN WORSE THERE IS STILL ANOTHER POSSIBLE VIRTUAL STATE AND THAT IS WHERE THE VIRTUAL PHOTON, OF SUFFICIENT ENERGY, CREATES A VIRTUAL PAIR, WHICH ANNIHILATE AND ULTIMATELY RETURN TO THE ELECTRON,



THIS LAST DIAGRAM DESCRIBES WHAT IS CALLED VACUUM POLARIZATION. I'LL COME BACK TO THIS PROCESS A LITTLE LATTER.

RIGHT NOW I WANT TO CONSIDER RADIATIVE CORRECTIONS WHICH ARE ONLY ACCURATE TO FIRST ORDER IN $e^2/\hbar c$, I.E. $\approx 1\%$. I HAVE THUS CHOSEN TO EXCLUDE ALL THE VIRTUAL STATES WHERE MORE THAN ONE PHOTON EXISTS. NONETHELESS I STILL HAVE A SERIES OF DIAGRAMS OF THE TYPE ON PAGE 45 WHICH MUST BE DEALT WITH. SINCE THAT IS GOING TO CREATE ENOUGH PROBLEM IN ITSELF I CHOOSE TO MAKE IT AS EASY AS POSSIBLE BY SIMPLIFYING THE ANALYSIS.

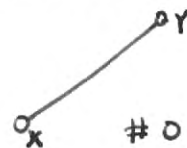
TO BEGIN THE ANALYSIS THE ELECTRON IS ASSUMED TO BE NEARLY FREE, I.E. $p^2 \neq m^2$ PRECISELY, BUT INSTEAD

$$p^2 = m^2 (1 + \epsilon)^2$$

WHERE $\epsilon = \frac{\hbar}{mT}$ FROM THE UNCERTAINTY PRINCIPLE. T IS THE INTERVAL BETWEEN SCATTERINGS. SINCE T IS LARGE, ϵ IS SMALL.

NOW CONSIDER THE ZERO ORDER DIAGRAM FOR DETERMINING THE AMPLITUDE TO PROPAGATE FROM X TO Y , I.E.,

$$\text{AMP}(0) = Y \frac{1}{p - m} X$$



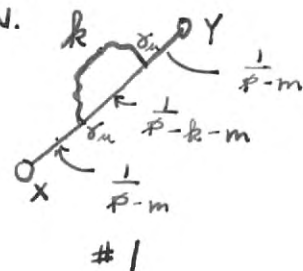
REWRITING

$$\text{AMP}(0) = Y \frac{p + m}{p^2 - m^2} X = Y \frac{(p + m) X}{2m^2 \epsilon}$$

TO ORDER ϵ .

THE NEXT ORDER DIAGRAM IS INTERESTING BECAUSE IT ALLOWS FOR THE EMISSION AND REABSORPTION OF A VIRTUAL PHOTON. BECAUSE WE MUST SUM OVER ALL POSSIBLE MOMENTA OF THE VIRTUAL PHOTON THE AMPLITUDE IS

$$\text{AMP}(1) = \int Y \frac{1}{\not{p}-m} \gamma_\mu \frac{1}{\not{p}-\not{k}-m} \gamma_\mu \frac{1}{\not{p}-m} X \frac{4\pi e^2 d^4 k}{k^2 (2\pi)^4}$$



IF WE DEFINE THE VARIABLE $B(p)$ TO BE

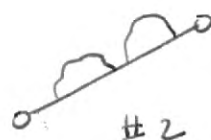
$$B(p) = \int \gamma_\mu \frac{1}{\not{p}-\not{k}-m} \gamma_\mu \frac{d^4 k}{k^2} \frac{(4\pi e^2)}{(2\pi)^4}$$

THEN WE CAN WRITE

$$\text{AMP}(1) = Y \frac{1}{\not{p}-m} B(p) \frac{1}{\not{p}-m} X$$

WE CAN NOW WRITE THE NEXT ORDER DIAGRAM AS

$$\text{AMP}(2) = Y \frac{1}{\not{p}-m} B(p) \frac{1}{\not{p}-m} B(p) \frac{1}{\not{p}-m} X$$



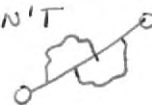
NOW WE HAVE A SERIES OF THESE AMPLITUDES WHICH MUST BE ADDED TO INCLUDE ALL POSSIBLE ~~for~~ CORRECTIONS IN e^2 . THUS TO SUM THE SERIES WE MAKE USE OF THE SENSATIONAL FACT THAT

$$\frac{1}{A+B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} - \dots$$

WHERE A AND B ARE NON-COMMUTING OPERATORS. THE SUM OF ALL THESE "BUBBLE" AS SOME THINK OF THEM IS JUST

$$Y \frac{1}{\not{p}-m-B(p)} X = \sum_{\text{All } e^2 \text{ terms}} \text{AMP}$$

AGAIN OUR APPROXIMATION IS TO ORDER e^2 AND WE HAVEN'T INCLUDED THE DIAGRAMS TO ORDER e^4 WHICH ARE LIKE THE ONE ON THE RIGHT.



IT WOULD BE NICE IF $B(p)$ WERE JUST A NUMBER BECAUSE IT WOULD THEN CORRESPOND TO A CORRECTION TO PROPAGATOR, I.E., THE MASS IS CHANGED BY A SMALL AMOUNT. PHYSICALLY, THIS MEANS THAT THE EXPERIMENTALLY MEASURED MASS REALLY INCLUDES THE EFFECT OF THE VIRTUAL PROCESSES JUST MENTIONED.

Thus B represents the correction to the theoretical mass due to the self energy of the particle. If we assume that $B(p)$ has the mathematical form of

$$B(p) = a(p) \not{p} + b(p)$$

Then the amplitude to get from X to Y is

$$X \rightarrow Y \frac{1}{(1-a)\not{p} - (m+b)} X = Y \frac{[(1-a)\not{p} + (m+b)] X}{(1-a)^2 p^2 - (m+b)^2}$$

or written differently as

$$\frac{Y}{\sqrt{1-a}} \frac{1}{(\not{p} - m')} \frac{X}{\sqrt{1-a}}$$

where $m' = \frac{m+b}{1-a} \approx m + b + ma$

and $b + ma$ is the correction to the mass.

The extra term $\sqrt{1-a}$ under Y and X are curious terms because they are interpreted to mean that each time the photon interacts it has its charge slightly changed so the coupling strength is modified. The change in mass and charge of the electron are referred to as mass and charge renormalization. The idea then is with a somewhat different mass and somewhat different coupling the mass of the electron is really known because you can work backward from the experimentally determinable mass, m' .

Well, not quite! You see $B(p)$ turns out to be an infinite integral. You can readily see that by observing that the integral is of the form $\int \frac{k^3 dk}{k^4} \propto \int \frac{dk}{k} \propto \ln k$ thus the integral is logarithmically divergent. What do we do? Certainly if $B = \infty$ the whole QED theory collapses. But since it is a good theory to explain the phenomena we see we would like to use it. We have to fix up the theory by modifying the rules of photon propagation.

LAMBDA THEORY for MASS RENORMALIZATION

Suppose THAT THE PROPAGATOR FOR A VIRTUAL PHOTON IS $1/k^2$ FOR SMALL VALUES OF k (I.E. CORRESPONDING TO WAVELENGTHS ON THE ORDER OF METERS, CENTIMETERS, MILLIMETERS} DOWN TO WAVELENGTHS OF 10^{-13} CM AND ENERGIES ON THE ORDER OF BEV'S). BEYOND THIS LIMIT WE WILL ASSUME THE PROPAGATOR FALLS OFF MORE RAPIDLY,

$$\frac{1}{k^2} \rightarrow \frac{1}{k^2} \left[\frac{-\Lambda^2}{k^2 - \Lambda^2} \right]$$

THIS CONVERGENCE FACTOR, CALL IT $C(k) = -\Lambda^2/k^2 - \Lambda^2$, IS CHOSEN TO BE RELATIVISTICALLY INVARIANT. FOR SMALL VALUES OF k^2 $C(k)$ IS ALMOST 1; FOR LARGE VALUES OF k^2 $C(k)$ GOES AS $1/k^2$ SO THE WHOLE PROPAGATOR GOES AS $1/k^4$. THE Λ FACTOR IS CHOSEN TO MAKE ALL THIS APPROXIMATION STUFF WORK. WE SHOULD NOW TRY TO EVALUATE BCP) USING THIS NEW PROPAGATOR.

THEREFORE WE WANT TO EVALUATE

$$B^\Lambda = 4\pi e^2 \int \gamma_\mu \frac{(\not{p} - \not{k} + m)}{(k^2 - 2p \cdot k + p^2 - m^2)} \gamma_\mu \frac{d^4 k}{k^2} \left[\frac{-\Lambda^2}{k^2 - \Lambda^2} \right]$$

USING THE RELATIONSHIPS THAT $\gamma_\mu \gamma_\mu = 4$ AND $\gamma_\mu \not{a} \gamma_\mu = -2a$ TO RID THE EQUATION OF γ_μ 'S AND ALSO NOTING TO REPLACE p^2 BY m^2 SINCE WE WANT THE MASS CORRECTION THEN

$$B^\Lambda = 4\pi e^2 \int \frac{-2(\not{p} - \not{k}) + 4m}{-2p \cdot k + k^2} \frac{d^4 k}{k^2} \left[\frac{-\Lambda^2}{k^2 - \Lambda^2} \right]$$

THIS INTEGRAL CAN BE EVALUATED TO BE

$$B^\Lambda = b + ma = \frac{3}{2} m \frac{e^2}{\pi} \left[3 \ln\left(\frac{\Lambda}{m}\right) + \frac{3}{4} \right] = \delta m$$

THE INTEGRAL NOW CONVERGES BUT IT DEPENDS ON Λ . AS Λ GETS BIGGER AND BIGGER, CORRESPONDING TO HIGHER AND HIGHER FREQUENCIES THE MASS CORRECTION GETS BIGGER LIKEWISE. THAT IS

$$M_{\text{EXPER.}} = M_{\text{THEORETICAL}} \left[1 + \frac{3e^2}{2\pi} \ln\left(\frac{\Lambda}{m}\right) \right]$$

THE PROBLEM IS YOU CAN'T TAKE THE LIMIT AS Λ GOES TO ∞ BECAUSE THE MASS CORRECTION BLOWS UP. IF Λ WERE REAL AND HAD A PHYSICAL BASIS, WE COULD UNDERSTAND HOW TO USE IT. BUT WE WORK BLIND.

EVEN IF Λ WERE VERY LARGE SAY ~ 100 BEV THE MASS CORRECTION IS ON THE ORDER OF 10%. IT SEEMS LIKE THE MASS CORRECTION IS INSENSITIVE TO THE HIGH FREQUENCY CUT OFF.

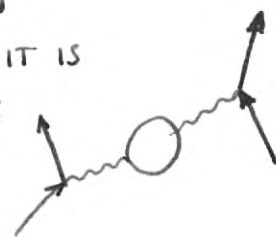
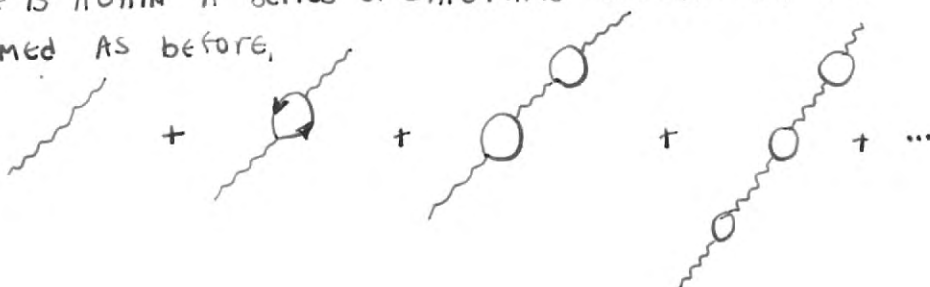
IN DEALING WITH HIGHER ORDER CORRECTIONS TO PROCESSES INVOLVING ELECTRON-PHOTON INTERACTION, LOGARITHMIC DIVERGENCES ARE THE WORSE KIND ENCOUNTERED, SO FAR THAT IS. HOWEVER, IF THE ANSWER TO EVERY PHYSICAL QUESTION IS EXPRESSED IN TERMS OF THE EXPERIMENTAL MASS, I.E., $M_{\text{EXP.}} = m_{\text{the.}} + \delta m$ THEN EXPANDED TO FIRST ORDER IN δm , THE COEFFICIENT OF $\ln \Lambda/m$ BECOMES EQUAL TO ZERO AND THE RESULT IS INSENSITIVE TO THE CUT OFF FREQUENCY; THE LIMIT AS $\Lambda \rightarrow \infty$ CAN BE TAKEN AND THE RESULT IS FINITE.

WE THUS HAVE ANOTHER RULE FOR QED. WHEN SOLVING PROBLEMS OF THIS TYPE: (1) PUT IN AN ARBITRARY CUT OFF FACTOR $C(k^2) = -\Lambda^2/k^2 - \Lambda^2$ AND OBSERVE THE SENSITIVITY TO Λ ; (2) EXPRESS EVERYTHING IN TERMS OF THE EXPERIMENTAL MASS; (3) TAKE THE LIMIT AS $\Lambda \rightarrow \infty$ AND GET THE CORRECT ANSWER.

WHILE THIS PROCEDURE DOES SEEM TO AGREE QUITE ACCURATELY WITH EXPERIMENTS, IT IS NOT CLEAR THAT IT IS A VALID ONE. THE ANSWER STILL COULD BE DIVERGENT AS WE CONSIDER HIGHER ORDER CORRECTIONS, I.E., WHERE 2 OR MORE VIRTUAL PHOTONS EXIST SIMULTANEOUSLY. THE THEORY IS VALID TO CORRECTIONS IN ORDER e^2 BUT THE SUM OF THE SERIES GOES AS $n! (\frac{e^2}{hc})^n$. WHEN $n \approx 137$ THEN $n!$ STARTS TO DOMINATE THE SERIES AND IT MAY DIVERGE.

CHARGE RENORMALIZATION, VACUUM POLARIZATION

THERE IS ANOTHER INFINITY WHICH WE SHOULD DEAL WITH AND IT IS ASSOCIATED WITH THE PROBABILITY OF CREATING A VIRTUAL PAIR: THERE IS AGAIN A SERIES OF DIAGRAMS WHICH MUST BE SUMMED AS BEFORE.



USING THE SAME TRICK AS BEFORE THE SERIES CAN BE SUMMED TO PRODUCE, IN EFFECT, A MODIFIED CHARGE e' COUPLING THE PHOTON TO THE ELECTRON. e' IS THE EXPERIMENTAL CHARGE MEASURED AND IT DIFFERS FROM THE THEORETICAL VALUE AS

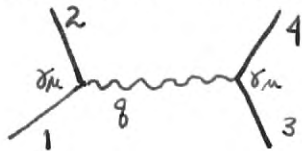
$$e_{\text{exp}} = \frac{e_{\text{theoretical}}}{\sqrt{1-\gamma}} = e'$$

WHERE γ APPROACHES A CONSTANT. e' IS CALLED THE RENORMALIZED CHARGE. THE INTERPRETATION OF THE CHARGE SHIFT IS THAT IT IS A SMALL CORRECTION TO THE COULOMB POTENTIAL. HOWEVER, A LOGARITHMIC DIVERGENCE STILL APPEARS SO WE EXPRESS THE ANSWER IN TERMS OF e' , TAKE THE LIMIT IN THE FIRST FEW ORDERS OF THE PERTURBATION. IT IS AMAZINGLY THAT WHILE ALL THESE TRICKS SEEM TO WORK AND GIVE THE RIGHT ANSWER WE DON'T UNDERSTAND WHY - THE PROBLEM IS TOO HARD TO SOLVE.

MORE ON CHARGE RENORMALIZATION

LAST TIME WE TALKED BRIEFLY ABOUT CHARGE RENORMALIZATION AND I'D LIKE TO DISCUSS IT NOW IN MORE DETAIL.

CONSIDER THE PROBLEM OF TWO ELECTRONS INTERACTING THROUGH THE COULOMB FIELD AND SCATTERING AT SMALL ANGLES



THE ELECTRONS ARE COULOMB COUPLED BY THE VIRTUAL PHOTON q WITH AN AMPLITUDE GIVEN BY:

$$4\pi e^2 (\bar{u}_4 \gamma_\mu u_3) \frac{1}{q^2} (\bar{u}_2 \gamma_\mu u_1)$$

THERE IS ANOTHER DIAGRAM, ONE OF MANY, FOR THE ABOVE INTERACTION WHEREIN A VIRTUAL PAIR IS PRODUCED, I.E.,



ON PAGE 50 ARE SOME OF THE OTHER POSSIBLE DIAGRAMS. TO COMPUTE THE CONTRIBUTION OF THIS DIAGRAM TO THE SCATTERING RATE WE HAVE TO FOLLOW THE ELECTRON AROUND THE CLOSED LOOP. WE START WITH $\sqrt{4\pi} e \gamma_\mu$ WHICH IS THE AMPLITUDE TO ANNIHILATE THE μ PHOTON; $1/(p-m)$ IS THE AMPLITUDE FOR THE ELECTRON TO PROPAGATE BETWEEN THE 2 PHOTON VERTICES; $\sqrt{4\pi} e \gamma_\nu$ IS THE AMPLITUDE TO EMIT A PHOTON OF POLARIZATION ν ; AND $1/(p-q-m)$ IS THE AMPLITUDE FOR THE ELECTRON TO PROPAGATE BACK TO ITS STARTING PLACE. THE TOTAL AMPLITUDE IS THEN

$$4\pi e^2 (\bar{u}_4 \gamma_\mu u_3) \frac{1}{q^2} \left[\int \text{spur} \left(\gamma_\mu \frac{1}{p-m} \gamma_\nu \frac{1}{p-q-m} \right) \frac{d^4 p}{(2\pi)^4} \right] \frac{1}{q^2} (\bar{u}_2 \gamma_\mu u_1)$$

THE INTEGRAL SUMS OVER ALL POSSIBLE MOMENTA p AND INITIAL STATES. A SIDE REMARK MAY BE IN ORDER TO EXPLAIN THIS POINT IN MORE DETAIL. THE INTEGRAL CONTAINS A SPUR WHICH MEANS WE SUM OVER ALL DIRECTIONS OF THE ELECTRON SPIN.

LET'S TAKE A LOOP WHERE WE START IN SOME STATE u_i AND END UP IN THAT STATE AFTER CIRCULATING AROUND THE LOOP. WE WANT TO EVALUATE

$$\sum_{\text{SPIN } u_i} (\bar{u}_i M u_i)$$



WHERE M IS THE MATRIX ELEMENT TO GET AROUND THE LOOP,

$$M = \gamma_u \frac{1}{p - q - m} \gamma_v \frac{1}{p - m}$$

ORDINARILY WHEN A MATRIX ~~IS~~ IS PROJECTED INTO $\bar{u} u$ AND SUMMED OVER ALL VALUES OF u , THE RESULT IS THE TRACE OR SPUR OF THE MATRIX, I.E.

$$\sum_{\text{ALL } u} (\bar{u} M u) = \text{Spur } M = \text{TRACE } M$$

WHERE TRACE M EQUALS THE SUM OF THE DIAGONAL ELEMENTS.

NOW SUPPOSE WE WANT THE SUM OVER 2 DIFFERENT SPINS. WHY NOT SUM OVER ALL u 'S AND SOMEHOW RESTRICT THE RESULT TO THE ~~FOUR~~ FOUR SOLUTIONS TO TWO. SINCE u MUST SATISFY $\not{p}u = mu$ AN ELECTRON AT REST MUST SATISFY $m\gamma_4 u = mu$ WHICH HAS THE SOLUTIONS

$$u = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{ELECTRON STATES}} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{POSITON STATES}}$$

WE WANT THE SOLUTIONS THAT SATISFY $\not{p}u = mu$ AND DON'T WANT THE SOLUTIONS THAT SATISFY $-\not{p}v = mv$. SO LET'S REWRITE THE \sum OVER SPINS OF u AS

$$\sum_{\text{SPIN } u} (\bar{u} M u) = \sum_{\text{ALL } 4 u\text{'S}} \frac{1}{2m} (\bar{u} M (\not{p} + m) u) = \text{Tr} \left[\frac{M(\not{p} + m)}{2m} \right]$$

THUS WHEN WE SUM OVER THE TWO NEGATIVE STATES WE GET ZERO.

SOME USEFUL PROPERTIES OF A SPUR ARE


$$\begin{aligned} \text{Sp} [1] &= 4 \\ \text{Sp} [\gamma_\mu] &= 0 \\ \text{Sp} [\gamma_i \gamma_j] &= 0 \\ \text{Sp} [\gamma_5] &= 0 \\ \text{Sp} [\gamma_i \gamma_i] &= -4 \end{aligned}$$

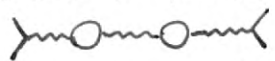
RETURNING TO THE INTEGRAL LET'S SEE WHAT HAPPENS WHEN WE TRY TO WORK IT OUT. CONSIDER THE INTEGRAL HAS A FORM OF THE FOLLOWING

$$\int_{SP} \left[\gamma_\mu \frac{1}{\not{p}-m} \gamma_\nu \frac{1}{\not{p}-\not{q}-m} \right] \frac{d^4p}{(2\pi)^4} = a \delta_{\mu\nu} + b q^2 \delta_{\mu\nu} + c q_\mu q_\nu + \dots$$

WE SUPPOSE THAT q IS SMALL ENOUGH THAT WE CAN MAKE A POWER EXPANSION IN q OF THE TERM $\frac{1}{\not{p}-m-q}$. WHEN WE DO THAT AND EVALUATE THE INTEGRAL WE GET AN AMPLITUDE LIKE

$$-(4\pi e^2)^2 (\bar{u}_2 \gamma_\mu u_1) \frac{1}{q^2} a \frac{1}{q^2} (\bar{u}_4 \gamma_\nu u_3)$$

BUT AS q GETS SMALL THIS TERM MAY DOMINATE THE  DIAGRAM WHICH IS TROUBLING SO WE MUST CONSIDER THE NEXT ORDER DIAGRAM,

 WHICH, WHEN EVALUATED PRODUCES

$$+(4\pi e^2)^3 (\bar{u}_2 \gamma_\mu u_1) \frac{1}{q^2} a \frac{1}{q^2} a \frac{1}{q^2} (\bar{u}_4 \gamma_\nu u_3)$$

NOW YOU KEEP DOING THIS AND FINALLY ADD UP ALL THE TERMS,

$$4\pi e^2 (\bar{u}_4 \gamma_\nu u_3) \left\{ \frac{1}{q^2} - \frac{4\pi e^2 a}{q^4} + \frac{(4\pi e^2 a)^2}{q^6} + \dots \right\} (\bar{u}_2 \gamma_\mu u_1) = 4\pi e^2 (\bar{u}_4 \gamma_\nu u_3) \frac{1}{q^2 - 4\pi e^2 a} (\bar{u}_2 \gamma_\mu u_1)$$

THIS IS INTERESTING BECAUSE IT SAYS THE LAW FOR PHOTON PROPAGATION IS WRONG AND REALLY THE PHOTON ~~MA~~ HAS A REST MASS $4\pi e^2 a$. WELL EXPERIMENTALLY NO EVIDENCE SUPPORTS THAT CONCLUSION SO a MUST BE ZERO. IF $a \neq 0$ THE Q.E.D WOULD NOT DESCRIBE NATURE RIGHT.

TERMS INVOLVING q_μ ARE ZERO SINCE $\bar{u}_2 \gamma_\mu u_1 = \text{CURRENT} = j_\mu$ AND THE DIVERGENCE OF THE CURRENT IS ZERO. SO WE WON'T WORRY ABOUT $q_\mu q_\nu$ TERMS. THAT LEAVES ONLY THE $b q^2 \delta_{\mu\nu}$ TERM TO WORRY ABOUT.

BY A SIMILAR SERIES EXPANSION AS ABOVE FOR THE a TERM WE GET THAT THE LOOP AMPLITUDE IS GIVEN BY

$$4\pi e^2 (\bar{u}_2 \gamma_\mu u_1) \frac{1}{q^2 - q^2 b (4\pi e^2)} (\bar{u}_4 \gamma_\nu u_3)$$

I CAN REWRITE THIS AS

$$4\pi e^2 j_\mu \frac{1}{q^2} \frac{1}{(1 + 4\pi e^2 b)} j_\nu$$

b HERE IS DEFINED AS A NEGATIVE QUANTITY. SO FINALLY WE CAN

WRITE $4\pi (e')^2 j_\mu \frac{1}{g^2} j_\mu$

WHERE THE NEW CHARGE $e' = \frac{e}{\sqrt{1 + 4\pi e^2 b}}$


THE COUPLING LAW THUS CHANGES SO THE CURRENTS INTERACT ACCORDING TO MAXWELL'S COULOMB LAW BUT WITH A MODIFIED CHARGE. WHENEVER WE QUOTE "e" WE REALLY MEAN e' BECAUSE WE CAN'T SOLVE THE INTEGRAL FOR b. IN ESSENCE THE CHARGE IS RENORMALIZED AND THE LAW SAYS TO PLEASE EXPRESS THE RESULTS IN TERMS OF e' BECAUSE e CAN'T BE DETERMINED. THE INTEGRAL FOR b CONTAINS A LOGARITHMIC DIVERGENT TERM WHICH REQUIRES AN ARBITRARY CUTOFF TO BE MADE IN ORDER TO OBTAIN A FINITE VALUE FOR b. THIS IS ANOTHER DILEMMA OF Q.E.D.

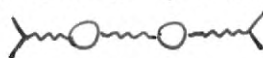
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48. AN INTRODUCTION TO WEAK INTERACTION

There is a class of interactions which does not obey the known laws of electrodynamics, gravity, or strong interactions. The interaction was observed in nuclei that undergo a decay such as,



This is called β -decay. Here a neutron decays into a proton by emitting an electron and antineutrino. The neutrino is a massless particle with spin $1/2$. Other types of β -decay are



The characteristics of these interactions are the following:

- (1). REACTION TIMES ON THE ORDER OF 10^{-10} SEC
- (2). RIGHT AND LEFT SYMMETRY IS ALWAYS VIOLATED
- (3). STRANGENESS IS VIOLATED

IN ORDER TO EXPLAIN THESE INTERACTIONS FERMI (~1934) PROPOSED THAT THEY OBEY THE SAME TYPE OF RULE AS ELECTRODYNAMICS WHICH MEANT THE TRANSITION RATE IS PROPORTIONAL TO $|M_{if}|^2$ WHERE M_{if} IS SOME MATRIX ELEMENT BETWEEN THE INITIAL AND FINAL STATE.

UNFINISHED TRANSCRIPTION FOLLOWS. THESE NOTES MAY NOT BE IN PROPER ORDER. THEY ARE INCLUDED HERE FOR TWO REASONS: 1) MAYBE THEY ARE IMPORTANT AND USEFUL TO SOMEONE AND 2) THESE ARE "RAW" LECTURE NOTES. SOME HAVE EXPRESS CURIOSITY OVER HOW I CAPTURED FEYNMAN WITHOUT A/V AID. TO GET THE TRANSCRIPTION CLOSE TO REAL TIME I HAD TO RUSH HOME AND DO THE TRANSCRIPTION THAT EVENING. (JUN 11/2012)