Solutions for MID-TERM
Problem 1.

Calculate the first and second-orders corrections to the energy eigenvalues of a linear harmonic oscillator with the cubic term $-\lambda\mu x^3$ added to the potential. Discuss the condition for the validity of the approximation.

The Hamiltonian of the perturbed system is $H = H^{(0)} + \lambda H^{(1)}$ where $H^{(0)} = \frac{1}{2m}p_x^2 + \frac{1}{2}kx^2$, $H^{(1)} = -\mu x^3$. The first-order correction to energy eigenvalues is given by

$$E_n^{(1)} = \langle n| - \mu x^3|n\rangle = -\mu\left(\frac{\hbar}{2m\omega}\right)^{3/2} \langle n|(a+a^\dagger)^3|n\rangle \ .$$

The expansion of $(a+a^\dagger)^3$ is

$$a^3 + a^2a^\dagger + aa^\dagger a + a^\dagger a^2 + a^{\dagger 2}a + a^\dagger aa^\dagger + aa^{\dagger 2} + a^{\dagger 3} \ .$$

In the above expansion each term has unequal powers of a and $a^\dagger$. Hence, $\langle n|(a+a^\dagger)^3|n\rangle = 0$ and $E_n^{(1)} = 0$. The first-order correction to the energy eigenvalues is thus 0. Next, calculate the second-order correction to E_n. We have

$$\begin{aligned} E_n^{(2)} &= \sum_{m\neq n} \frac{\left|\langle n|H^{(1)}|m\rangle\right|^2}{E_n^{(0)} - E_m^{(0)}} \\ &= \frac{\mu^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3 \sum_{m\neq n} \frac{\left|\langle n|(a+a^\dagger)^3|m\rangle\right|^2}{(n-m)} \ . \end{aligned}$$

Consider the term $\langle n|(a+a^\dagger)^3|m\rangle$. It is expanded as

$$\langle n|a^3 + a^2a^\dagger + aa^\dagger a + a^\dagger a^2 + a^{\dagger 2}a + a^\dagger aa^\dagger + aa^{\dagger 2} + a^{\dagger 3}|m\rangle \ .$$

We evaluate each term in the above integral. We obtain

$$\begin{aligned} \langle n|a^3|m\rangle &= \langle n|a^2\sqrt{m}|m-1\rangle \\ &= \langle n|a\sqrt{m(m-1)}|m-2\rangle \\ &= \langle n|\sqrt{m(m-1)(m-2)}|m-3\rangle \\ &= \sqrt{m(m-1)(m-2)}\,\delta_{n,m-3} \\ \langle n|a^2a^\dagger|m\rangle &= \langle n|a^2\sqrt{m+1}|m+1\rangle \\ &= (m+1)\sqrt{m}\,\delta_{n,m-1} \\ \langle n|aa^\dagger a|m\rangle &= m\sqrt{m}\,\delta_{n,m-1} \\ \langle n|a^\dagger a^2|m\rangle &= (m-1)\sqrt{m}\,\delta_{n,m-1} \\ \langle n|a^{\dagger 2}a|m\rangle &= m\sqrt{m+1}\,\delta_{n,m+1} \\ \langle n|a^\dagger aa^\dagger|m\rangle &= (m+1)\sqrt{m+1}\,\delta_{n,m+1} \\ \langle n|aa^{\dagger 2}|m\rangle &= (m+2)\sqrt{m+1}\,\delta_{n,m+1} \\ \langle n|a^{\dagger 3}|m\rangle &= \sqrt{(m+1)(m+2)(m+3)}\,\delta_{n,m+3} \ . \end{aligned}$$

Then

$$\begin{aligned} \langle n|(a+a^\dagger)^3|m\rangle &= \sqrt{m(m-1)(m-2)}\,\delta_{n,m-3} \\ &\quad +3m^{3/2}\,\delta_{n,m-1} + 3(m+1)^{3/2}\,\delta_{n,m+1} \\ &\quad +\sqrt{(m+1)(m+2)(m+3)}\,\delta_{n,m+3} \ . \end{aligned}$$

In the summation in the expression for $E_n^{(2)}$ the nonzero contribution of $\langle n|(a+a^\dagger)^3|m\rangle$ comes from the cases $m = n+3$, $n+1$, $n-1$ and $n-3$. Then

$$\begin{aligned} E_n^{(2)} &= \frac{\mu^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3 \left[\frac{(n+1)(n+2)(n+3)}{-3} + \frac{9(n+1)^3}{-1} \right. \\ &\qquad \left. + \frac{9n^3}{1} + \frac{n(n-1)(n-2)}{3}\right] \\ &= -\frac{\mu^2\hbar^2}{8m^3\omega^4}\left(30n^2 + 30n + 11\right) \ . \end{aligned}$$

Since $E_n^{(2)}$ is negative, all the energy eigenvalues are reduced. The amount of reduction increases with n. This is because due to the cubic term the potential flattens for large x.

The ratio of the change in energy due to the cubic term is

$$\frac{E_n^{(2)}}{E_n^{(0)}} = -\frac{\mu^2\hbar}{4m^3\omega^5}\frac{\left(30n^2+30n+11\right)}{(2n+1)} .$$

A condition for the validity of the perturbation theory is that the above ratio must be small. This requires both $\mu^2\hbar/(m^3\omega^5)$ and $\alpha = (30n^2 + 30n + 11)/4(2n+1)$ to be small. α is small provided n is limited to a low number. We note that for sufficiently large x, the potential $V(x)$ is negative and below the origin. Hence, a state with energy below the maximum, say, A is not truly a bound state but has a small probability of tunneling out to the right. For low lying states this probability is negligible. But for higher states the perturbation theory breaks down.

Problem 3.

A one-dimensional linear harmonic oscillator is acted upon by the force $F(t) = \dfrac{F_0\tau/\omega}{\tau^2+t^2}$, $-\infty < t < \infty$. At $t = -\infty$, the oscillator is in the ground state. Using the time-dependent perturbation theory to first-order, calculate the probability that the oscillator is found to be in the excited state at $t = \infty$.

The transition coefficient $a_1^{(1)}(t)$ for the given problem is

$$\begin{aligned} a_1^{(1)}(t) &= -\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t'}\left\langle 1\left|H^{(1)}\right|0\right\rangle \mathrm{d}t' \\ &= \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t'}\left\langle 1\left|x\right|0\right\rangle \frac{F_0\tau/\omega}{\tau^2+t'^2}\,\mathrm{d}t' \\ &= \frac{\mathrm{i}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2}(F_0\tau/\omega)\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t'}}{\tau^2+t'^2}\,\mathrm{d}t' . \end{aligned}$$

The integral in the above equation can be evaluated using contour integration. Its value is $(\pi/\tau)\mathrm{e}^{-\omega\tau}$. Then

$$a_1^{(1)}(t) = \frac{\mathrm{i}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\frac{F_0\pi}{\omega}\,\mathrm{e}^{-\omega\tau} .$$

and hence

$$\left|a_1^{(1)}(t)\right|^2 = \frac{F_0^2\pi^2}{2m\hbar\omega^3}\,\mathrm{e}^{-2\omega\tau} .$$

The time $\tau \to \infty$ corresponds to turning the perturbation slowly, that is, $\omega\tau \gg 1$. Hence, the transition probability vanishes. The other limit $\omega\tau \to 0$ corresponds to the application of an impulsive perturbation with $\lim\limits_{\tau\to 0}\dfrac{\tau}{\pi(t^2+\tau^2)} = \delta(t)$. Therefore, for $\tau \to 0$, $\left|a_1^{(1)}(t)\right|^2 = (F_0^2\pi^2)/(2m\hbar\omega^3)$.

Problem 4.

A particle of mass m is acted on by the three-dimensional potential $V(r) = -V_0 \mathrm{e}^{-r/a}$ where $\hbar^2/(V_0 a^2 m) = 3/4$. Use the trial function $\mathrm{e}^{-r/\beta}$ to obtain a bound on the energy.

The normalization condition gives $N = \sqrt{1/(\pi\beta^3)}$. Since V is independent of θ and ϕ

$$\begin{aligned} \langle E \rangle &= -4\pi N^2 \frac{\hbar^2}{2m} \int_0^\infty \mathrm{e}^{-r/\beta} r^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} \right) \mathrm{e}^{-r/\beta} \, \mathrm{d}r \\ &\quad -4\pi N^2 V_0 \int_0^\infty \mathrm{e}^{-2r/\beta} \mathrm{e}^{-r/a} r^2 \, \mathrm{d}r \; . \end{aligned}$$

Carrying out the differentiation the above integral we get

$$\begin{aligned} \langle E \rangle &= -\frac{4\pi\hbar^2 N^2}{2m\beta^2} \int_0^\infty \mathrm{e}^{-2r/\beta} r^2 \, \mathrm{d}r + \frac{8\pi\hbar^2 N^2}{2m\beta} \int_0^\infty \mathrm{e}^{-2r/\beta} r \, \mathrm{d}r \\ &\quad -4\pi N^2 V_0 \int_0^\infty \mathrm{e}^{-\left(\frac{2}{\beta}+\frac{1}{a}\right)r} r^2 \, \mathrm{d}r \; . \end{aligned}$$

That is,

$$\begin{aligned} \langle E \rangle &= -\frac{\pi\hbar^2 N^2}{2m\beta^2}\frac{2\beta^3}{8} + \frac{8\pi\hbar^2 N^2}{2m\beta}\frac{\beta^2}{4} - \frac{8\pi N^2 V_0}{\left(\frac{2}{\beta}+\frac{1}{a}\right)^3} \\ &= \frac{\hbar^2}{2m\beta^2} - \frac{8V_0}{\left(2+\frac{\beta}{a}\right)^3} \; . \end{aligned}$$

$\partial\langle E\rangle/\partial\beta = 0$ gives

$$\frac{32}{\left(2+\frac{\beta}{a}\right)^4} = \frac{a^3}{\beta^3} \; .$$

If $\beta/a = 2$ the above equation is satisfied. Therefore, $\beta = 2a$. Then $\langle E \rangle = -V_0/32$.

Problem 5

Calculate the differential cross-section for a central Gaussian potential $V(r) = (V_0/\sqrt{4\pi})\mathrm{e}^{-r^2/4a^2}$ under Born approximation.

Under the Born approximation

$$
\begin{aligned}
f &= -\frac{2mV_0}{s\hbar^2\sqrt{4\pi}} \int_0^\infty r\sin(sr)\,\mathrm{e}^{-r^2/(4a^2)}\,\mathrm{d}r \\
&= \frac{2mV_0}{s\hbar^2\sqrt{4\pi}} \frac{\partial}{\partial s} \int_0^\infty \cos sr\,\mathrm{e}^{-r^2/4a^2}\,\mathrm{d}r \\
&= \frac{mV_0}{s\hbar^2\sqrt{4\pi}} \frac{\partial}{\partial s} \int_{-\infty}^\infty \cos(sr)\,\mathrm{e}^{-r^2/4a^2}\,\mathrm{d}r\ .
\end{aligned}
$$

Writing $\cos(sr) = (\mathrm{e}^{\mathrm{i}sr} + \mathrm{e}^{-\mathrm{i}sr})/2$ and defining $x = (r/2a) - \mathrm{i}sa$, $y =$

$(r/2a) + \mathrm{i}sa$ and using the result $\int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$ we get

$$
\begin{aligned}
f &= \frac{mV_0 a}{s\hbar^2\sqrt{4\pi}} \frac{\partial}{\partial s} \mathrm{e}^{-s^2a^2} \left[\int_{-\infty}^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{-\infty}^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y\right] \\
&= \frac{mV_0 a}{s\hbar^2} \frac{\partial}{\partial s} \mathrm{e}^{-s^2a^2} \\
&= -\frac{2mV_0a^3}{\hbar^2} \mathrm{e}^{-s^2a^2}\ .
\end{aligned}
$$

Then

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2 = \frac{4m^2V_0^2a^6}{\hbar^4}\,\mathrm{e}^{-2s^2a^2}\ .
$$

Problem 6

Estimate the ground state of the infinite-well (one-dimensional box) problem defined by

$$V = \begin{cases} 0, & \text{for } |x| < L \\ \infty, & \text{for } |x| > L, \end{cases}$$

using the trial eigenfunction $\phi = |L|^\alpha - |x|^\alpha$ with α the trial parameter and compare it with the exact energy value.

We obtain

$$\begin{aligned}
\langle E \rangle &= \frac{-\dfrac{\hbar^2}{2m}\displaystyle\int_{-L}^{L}\left(\phi^* \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)\mathrm{d}x}{\displaystyle\int_{-L}^{L}\phi^*\phi\,\mathrm{d}x} \\
&= \frac{\dfrac{\hbar^2}{2m}\alpha(\alpha-1)\displaystyle\int_0^L (L^\alpha - x^\alpha)\,x^{\alpha-2}\,\mathrm{d}x}{\displaystyle\int_0^L \left(L^{2\alpha} - 2L^\alpha x^\alpha + x^{2\alpha}\right)\mathrm{d}x} \\
&= \frac{(\alpha+1)(2\alpha+1)}{2\alpha-1}\left(\frac{\hbar^2}{4mL^2}\right).
\end{aligned}$$

From $\partial\langle E\rangle/\partial\alpha = 0$ we get $\alpha = (1 \pm \sqrt{6})/2$. Since α has to be positive for physically acceptable solution we choose $\alpha = (1+\sqrt{6})/2 \approx 1.72$. Then using $E_{\text{exact}} = \hbar^2\pi^2/(8mL^2)$ we obtain

$$\langle E \rangle = \frac{2.72 \times 4.44 \times 2 \times E_{\text{exact}}}{2.44 \times \pi^2} = 1.003 E_{\text{exact}}\,.$$

The percentage of error is 0.3%.