Dirac Equation.

FOR FERMIONS!

$i\hbar\frac{\partial}{\partial t} \to E \Leftrightarrow H = c\,\alpha\cdot\bar{p} + \beta mc^2$, $p = -i\hbar\frac{\partial}{\partial r}$ $\to \nabla$

The corresponding wave equation is given by:

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\,(\alpha\cdot\nabla)\psi + \beta mc^2\psi$$

1. For a free particle H is independent of space and time. α and β are time independent.
2. H is linear in $\bar{p}$, it means α and β are p-independent.

E given by H by satisfy to relativistic energy-momentum $E^2 = p^2c^2 + m^2c^4$ or

$$\boxed{E = c\sum_{i=1}^{3}\alpha_i p_i + \beta mc^2}$$

$$E^2 = c^2\sum_i \alpha_i^2 p_i^2 + c^2\sum_{i,j\neq i}\sum_j(\alpha_i\alpha_j + \alpha_j\alpha_i)p_ip_j + mc^3\sum_i(\alpha_i\beta + \beta\alpha_i)p_i + m^2c^4\beta^2 \Rightarrow$$

compare to $E^2 = \bar{p}^2c^2 + m^2c^4$

$$\begin{cases} \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad i\neq j \\ \alpha_i\beta + \beta\alpha_i = 0 \\ \alpha_i^2 = \beta^2 = 1 \end{cases}$$

those cannot be just scalars

so an object like matrixes or higher order.

if α_i and β are matrixs then

$$\alpha_i\beta + \beta\alpha_i = 0 \quad \text{and} \quad \alpha_i^2 = \beta^2 = I \Rightarrow$$

eigenvalues $= \pm 1$

$$\alpha_i = \alpha_i\cdot I = \alpha_i\beta^2 \equiv \alpha_i\beta\beta$$

Next from $\alpha_i\beta + \beta\alpha_i = 0 \Rightarrow \alpha_i\beta = -\beta\alpha_i \;|\cdot\beta$ $(\beta\beta = I)$

$$\alpha_i = -\beta\alpha_i\beta$$

and $\longrightarrow$

Recal trace $ABC = Tr\, BCA$ we get

$$tr\,\alpha_i = tr\,\alpha_i \beta\beta = -\,tr\,\underbrace{\beta\beta}_{1}\alpha_i = -tr\,\alpha_i \Rightarrow$$
$$tr\,\alpha_i = 0$$

so this means that $+1$ and -1 should occur even number of times.

e.g. For $n=2$ those are Pauli matrices.

$$\sigma_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

e.g. for $n=4$ we need $\alpha_1\ \alpha_2\ \alpha_3$ and β (they must be hermitian).

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Lets rewrite the equation in the covariant form:

$$i\hbar \frac{\partial \psi}{\partial t} + i\hbar c \sum_{k=1}^{3} \alpha_k \frac{\partial \psi}{\partial x_k} - \beta m c^2 \psi = 0$$

or
$$\beta \frac{\partial \psi}{\underbrace{\partial(ict)}_{x_4}} - \sum \underbrace{i\beta\alpha_k}_{\gamma_k} \frac{\partial \psi}{\partial x_k} + \overset{\beta.\beta = I}{\frac{mc}{\hbar}} = 0$$

or
$$\sum_{k=1}^{4} \gamma_k \frac{\partial \psi}{\partial x_k} + \frac{mc}{\hbar}\psi = 0$$

$$\begin{cases} x_4 = ict \quad \gamma_4 = \beta \\ \gamma_k = -i\beta\alpha_k \quad k=1,2,3 \end{cases}$$

if you assume the Einstein rules the double index means summation we get

$$\boxed{\left(\gamma_k \frac{\partial}{\partial x_k} + mc/\hbar\right)\psi = 0}$$

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Read Solved Problem 2, p 370.

Here is the interesting question: <u>What about probability density</u>.

$$\psi^\dagger \times \left| \; i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c \alpha \cdot \nabla\psi + mc^2 \beta \psi \; \right|$$

$$i\hbar \psi^\dagger \frac{\partial \psi}{\partial t} = -i\hbar c\, \psi^\dagger \alpha \cdot \nabla \psi + mc^2 \psi^\dagger \beta \psi$$

$-$ and multiplying its hermitian version from the right by ψ

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} \psi = i\hbar c\, \nabla \psi^\dagger \cdot \alpha \psi + mc^2 \psi^\dagger \beta \psi$$

$$\frac{\partial}{\partial t} \underbrace{(\psi^\dagger \psi)}_{\rho} + c \nabla \cdot \underbrace{(\psi^\dagger \alpha \psi)}_{J} = 0$$

ρ ← like in NRQM!

$$\frac{\partial}{\partial t} \rho + \nabla J = 0$$

$\rho = \psi^\dagger \psi$ which ≥ 0, and α is positive and so is $J > 0$! much better than K-G.

Recal that:

$$J^k = (J, ic\rho) \Rightarrow \partial_k J^k = 0$$

<u>What about the solution of D.E.?</u>

<u>PLANE WAVE SOLUTION</u>:

Lets go back to the D.E. for free particle.

$$(c\alpha \cdot p + \beta mc^2) = E\psi$$

α and β are 4×4 matrices. and this means that the solution should be 4×1 matrix. or the 4-component function.

lets seek the solution in this form:

$$\psi(\bar{r},t) = N u(p)\, e^{i(p\cdot r - Et)/\hbar}$$

here we assume that u is linear in p.

$$(c\alpha\cdot p + \beta mc^2)\cancel{N} u(p)\, \cancel{e^{ipx}} = E\cancel{N} u(p)\, \cancel{e^{ipx}}$$

$$(c\alpha\cdot p + \beta mc^2)\, u(p) = E\, u(p)$$

and $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ conventionally it can be written as:

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \quad v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad w = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

if we explicety write down the equation in the matrix form:

$$E\begin{pmatrix} v \\ w \end{pmatrix} - mc^2 \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{=\beta} \begin{pmatrix} v \\ w \end{pmatrix} = c\begin{pmatrix} 0 & \sigma\cdot p \\ \sigma\cdot p & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}$$

it breaks down into 2 equations:

$$\begin{cases} (E - mc^2)v = c(\sigma\cdot p)w \\ (E \underset{-(-I)}{+} mc^2)w = c(\sigma\cdot p)v \end{cases}$$

now replace p by $p = -i\hbar\nabla$

we get a set of 2 diff. equations. for v, w and E

① Solution: Eigenvalue E

$$(E - mc^2)v = c(\sigma\cdot p)w \;|\times (E + mc^2)$$

$$(E^2 - m^2c^4)v = c^2(\sigma\cdot p)^2 v$$

(we replace RHS by $(E+mc^2)\cdot w$ $c(\sigma\cdot p)v$.

By using identity $(\sigma\cdot A)(\sigma\cdot B) = A\cdot B + i\sigma\cdot(A\times B)$

we get : $(\sigma\cdot\bar{p})^2 = \bar{p}^2$

then we can rewrite:

$$(E^2 - m^2c^4)\,v = c^2(\sigma\cdot p)^2 v$$

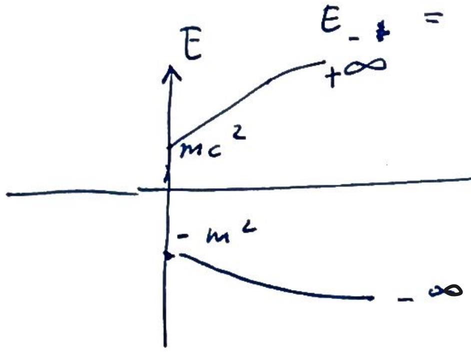
$$(E^2 - m^2c^4)\,v = c^2p^2v \qquad \underline{\underline{\quad}} = p^2$$

$$(E^2 - m^2c^4 - c^2p^2)\,v = 0 \qquad \text{or}$$

$$\boxed{E_\pm = \pm\sqrt{p^2c^2 + m^2c^4} \qquad \text{or } E_- = -E_+}$$

if a particle is in the rest $|p| = 0$

$$E_- = -mc^2 \qquad E_+ = mc^2$$



$E_+ - E_- = 2mc^2 \equiv$ gap!

There are no states inside $\pm mc^2$

Also there is no ground state!

(since spectrum extends to $-\infty$)

② Solutions: Eigenfunction $u(p)$

Clearly we need to determine v, w.

We substitude E_+ and E_- into the D.E.

when $|p| = 0$

i.e. if $\underline{E_+ = mc^2}$: $(\underbrace{E - mc^2}_{=0})v = c(\sigma\cdot p)w$

it means we need to use E_- here

the same for $(E + mc^2)$ ....

we use E_+ as the solution for E

$$(E_- - mc^2)v = c(\sigma\cdot p)w \Rightarrow \boxed{\begin{aligned} v &= -\frac{c\,\sigma\cdot p}{\underbrace{E_+ + mc^2}_{E_+ = -E_-}}\,w \\ w &= \frac{c\,\sigma\cdot p}{E_+ + mc^2}\,v \end{aligned}}$$

here is the problem:

recall we assumed tha $u(p)$ is linear in p so if v is linear in p w should be <u>non-linear</u> in p

From this we conclude that v can be taken as p-independent = const.

Similarly for v. normalized

e.g. v $\cancel{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\cancel{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

boring good

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Lets compute $\sigma \cdot \vec{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$

$$= \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix} \qquad p_\pm = p_x \pm i p_y$$

$$w = \frac{c}{E_+ + mc^2} \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix} \overset{v}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so we end up

Finally:

For E_+

① $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad w = \frac{c}{E_+ + mc^2} \begin{pmatrix} p_- \\ -p_z \end{pmatrix} \quad u = \begin{pmatrix} v \\ w \end{pmatrix}$

$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad w = \frac{c}{E_+ + mc^2} \begin{pmatrix} p_z \\ +p_+ \end{pmatrix} \quad u = \begin{pmatrix} v \\ w \end{pmatrix}$

For E_-

$w = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v = -\frac{c}{E_+ + mc^2} \begin{pmatrix} p_- \\ -p_z \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v = -\frac{c}{E_+ + mc^2} \begin{pmatrix} p_z \\ p_+ \end{pmatrix}$

Rewrite Dirac equation in components:

e.g. $v_1 = 0 \quad v_2 = 1 \quad v_3 = \frac{c}{E_+ + mc^2} p_-$

$v_4 = -\frac{c}{E_+ + mc^2} p_z$

and so on.

The normalization can be achieved as:

$|N|^2 (v_1^* v_1 + \ldots v_4^* v_4) = 1 =$

$= N^2 \left[1 + \frac{c^2 p^2}{(E_+ + mc^2)^2} \right] \Rightarrow$

$N = \frac{1}{\sqrt{1 + \frac{cp^2}{(E_+ + mc^2)^2}}}$

What's the meaning of 2 solutions? for E_+ and E_-

A: Dirac eqn. works for $s = 1/2$ fermions giving us two spin orientations for a given p and E.

More on this is in the next section ⇒

Recall the main problem with K-G.

QFT version of Dirac eqn.

A problem caused by $\partial^2 + m^2$ term in K-G = negative probability etc was caused by the fact that we have the second-order in derivatives.

One of the ideas can we try:

$$(\partial^2 + m^2) = (\sqrt{\partial^2} + im)(\sqrt{\partial^2} - im) \; ??$$

$\sqrt{\partial^2}$ is not defined — BAD IDEA?

Dirac "invented" new math. instead of giving up.

1) Define a new 4-vector γ^μ

$$(\gamma^0)^2 = 1 \quad (\gamma^i)^2 = -1 \quad i = 1,2,3$$

2) The components of the 4-vector are not numbers — they anti commute.

i.e. $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{ \; \} = 0$

or all together (1) + 2)

$$\{\gamma^\mu \gamma^\nu\} = 2 g^{\mu\nu}$$

3) New math — Clifford algebra

lets introduce a new symbol

$\not{a}$ - a-slash.

$$\not{a} = \gamma^\mu a_\mu \Rightarrow \not{\partial}^2 = (\gamma^\mu \partial_\mu)^2$$

$$= \left(\sum_{i=1}^{4} \gamma^i \frac{\partial}{\partial x_i}\right)^2 = \gamma_0^2 \left(\frac{\partial}{\partial t}\right)^2 + \gamma^{1\,2} \frac{\partial}{\partial x}^2 + \ldots$$

use $\{.\} = 2g^{\mu\nu}$

$$= \; = \; = \partial^2 \qquad \text{so} \quad \boxed{\not{\partial}^2 = \partial^2}$$

$$(\partial^2+m^2) = (\not\partial^2+m^2) = (\not\partial - im)(\not\partial + im)$$

Consider
$$\boxed{\begin{gathered}(\not\partial + im)\psi(x) = 0 \\ (i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0\end{gathered}}$$
← standard Dirac eqn.

interestingly we can write the equation in terms of p_μ $\quad p_\mu = i\partial_\mu = i(\frac{\partial}{\partial t}, \nabla)$

$$(\underbrace{i\gamma^{\mu}\partial_{\mu}}_{\not p} - m)\psi(x) = 0$$

or

$$\boxed{(\not p - m)\psi(x) = 0}$$

this is overly compact form of Dirac eqn.

Massless particles

Left - right handed wave function.

we alredy derived a set of γ^μ (4×4 matrices) at the begining of this lecture.

$$(\gamma^0 p^0 - \gamma\cdot p - m)\psi(x) = 0$$

$p^0 = i\partial^0 \quad p = i\nabla$

and 4 component vector $\psi = \begin{bmatrix}\psi_L \\ \psi_R\end{bmatrix}$

$$\left[\begin{pmatrix}0 & p^0 \\ p^0 & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma p \\ -\sigma p & 0\end{pmatrix} - \begin{pmatrix}m & 0 \\ 0 & m\end{pmatrix}\right]\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0$$

or
$$\begin{cases}(p_0 - \sigma\cdot p)\psi_R = m\psi_L \\ (p_0 + \sigma\cdot p)\psi_L = m\psi_R\end{cases}$$

Consider $m = 0$
$$\begin{cases}(p_0 - \sigma p)\psi_R = 0 \\ (p_0 + \sigma p)\psi_L = 0\end{cases}$$

So we have a separate equation for each component Ψ_L and Ψ_R.
So according to Dirac we should have 2 kinds of massless particles left and righ handed.

Definition of Left and Right handidness.

How can we define the handidness or helicity or chirality of the wave function?
we need to have 1) a chirality operator
2) act by it on the w.f.

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

then $\Psi(x) = \begin{pmatrix} \Psi_L \\ 0 \end{pmatrix}$ $\quad \gamma^5 \Psi(x) = -\Psi(x)$

and for $\begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}$ $\quad \gamma^5 \Psi(x) = +\Psi(x)$

To extract a piece of the wave function which is only left- or right-handed use the following projection operators

$$P_L = \frac{1-\gamma^5}{2} \qquad P_R = \frac{1+\gamma^5}{2}$$

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So ψ_L and ψ_R are the eigenstates of Dirac massless electron with $s = 1/2$

and L dirac particle never changes into right handed particles!

Also since $p^0 \psi = E \psi$
$= |p|$ ← for massless particle!

$\sigma \cdot p \, \psi_R = |p| \psi_R$ or

$$\psi_R = \frac{\sigma \cdot p}{|p|} \quad \text{and so is} \quad \psi_L = -\frac{\sigma \cdot p}{|p|} \psi_L$$

eigenstates of the helicity operator:

$$\hat{h} = \frac{\sigma \cdot \hat{p}}{|p|} \quad \text{with eigenvalues } \pm 1$$

Note: Helicity and chirality are the same for massless particles only.

In general: helicity tell us if p and s are parallel or antiparallel, it depends on the reference frame. By changing the frame we can reverse helicity of a massive particle.

However, in the case of massive particle is coupled by the mass
at the rest frame:

$$i \partial_0 \psi_R = m \psi_L \qquad i \partial_0 \psi_L = m \psi_R$$

What about Energy for a massive particle?

From $\begin{cases}(p_0 - \sigma p)\,\psi_R = m\psi_L \\ (p_0 + \sigma p)\,\psi_L = m\psi_R\end{cases}$

$$(p_0 + \sigma p)(p^0 - \sigma p)\,\psi_R = m^2 \psi_R$$

or $[p_0^2 - p^2] = m^2 \Rightarrow E_p = \pm (p^2 + m^2)^{1/2}$

and still has the negative energy states.

What about antiparticles?

Writing $E = -|p^0|$ we get

$$\begin{cases}(-|p^0| - \sigma p)\,\psi_R = 0 \\ (-|p^0| + \sigma p)\,\psi_L = 0\end{cases}$$ antiparticles

which means $\frac{\sigma p}{|p^0|}\psi_R = -\psi_R$ ← neg. helicity

$\frac{\sigma p}{|p^0|}\psi_L = \psi_L$ ← positive helicity

Finally, the 4-component Dirac w.f. have ψ_L and ψ_R

$$\psi(x) = \begin{pmatrix}\psi_L(x) \\ \psi_R(x)\end{pmatrix}$$ ← 4-component Called Dirac spinors

2-component Weil spinors

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Dirac and Weyl spinors.

Lets study ψ_R and ψ_L

1) Since particles and anti particles really both have positive energy solutions the convention to call them positive & negative

frequency solutions.

+ freq. for particles
− for antiparticles

$$\text{Particles: } u(p)e^{-ipx} = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix} e^{-ipx}$$

$$\text{anti particles: } v(p)e^{ipx} = \begin{pmatrix} v_L \\ v_R \end{pmatrix} e^{ipx}$$

note here we have $p^0 = E > 0$

The 4-component objects $u(p)$ and $v(p)$ are momentum space Dirac fermions:

$$(\not{p} - m)\, u(p) = 0$$
$$(-\not{p} - m)\, v(p) = 0$$

$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$ are momentum space Weyl fermions

Lets move to the rest frame where $p^\mu = (m, 0)$

$$\begin{pmatrix} -m & m \\ m & -m \end{pmatrix} \begin{pmatrix} u_L(p^0) \\ u_R(p^0) \end{pmatrix} = 0 \Rightarrow$$

$$u(p^0) \equiv \begin{pmatrix} u_L \\ u_R \end{pmatrix} = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

factor included for convinience

where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ a 2-component column vector
with $\xi^{+}\xi = 1$ = often called spinor

We can repeat the same for antiparticles

$$v(p^0) = \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix} \quad \text{with } \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Solutions to Dirac eq. has two degrees of freedom : 2-comp. ξ for particles
2 - comp. η for antiparticles

Next recall that in the rest frame $S = \frac{1}{2}\sigma$
For spins along z $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \uparrow$ $\quad \downarrow \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
those are the basis states.

For antiparticles: $\uparrow \eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \downarrow \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\hat{S}_z \eta = -\frac{1}{2}\eta$$

So for antiparticle with physical spin up $\uparrow$ the
we have the projection $-\frac{1}{2}$!! weird.

And for spin down $S_z = +\frac{1}{2}$!

For massless antiparticles with $h = +1$ and along p $\sigma\eta = +\eta$ Meaning again
$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which corresponds to physical spin down.

THE END OF RQM.