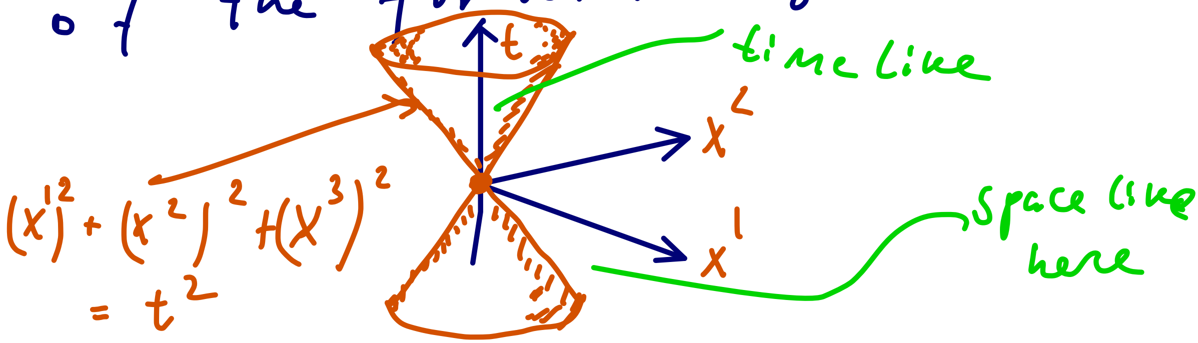
THE DEATH OF QM BY RELATIVISM.

First a reminder:
in the Heisenberg picture
if you know the state at $t=0$
$\psi(0) \Rightarrow \psi(t) = \hat{U}(t,0)\psi(0) =$
$= e^{-iHt}\psi(0)$

and $\dfrac{d\hat{O}(t)}{dt} = \dfrac{1}{i\hbar}[\hat{O}(t), H]$

Now we show something strange.
Lets calculate the probability
for a particle to travel outside
of the forward light cone



So we start at $t=0$ $\bar{x}=0$ and travel to $\bar{x}$ at the $t>0$ faster than light or $|\bar{x}|>t$ and the interval is space like

if this amplitude of probability $\neq 0$ IT MEANS THE DEATH OF Q.M.

Here it is:

$$A = \langle \bar{x} | e^{-iHt} | \bar{x}=0 \rangle$$

Let work in the momentum basis $H|p\rangle = E_p|p\rangle$ and

$$E_p = \sqrt{\bar{p}^2+m^2}$$

also $\langle x|p\rangle = e^{i\bar{p}\cdot\bar{x}} \cdot \frac{1}{(2\pi)^{3/2}}$

$$\langle x | e^{-iHt} | x=0 \rangle =$$

$$= \int d^3p \, \langle x | e^{-iHt} \underbrace{|p\rangle\langle p|}| x=0 \rangle$$

$$= \int d^3p \, \langle x | p \rangle \, e^{-iE_p t} \frac{1}{(2\pi)^{3/2}} e^{-i \cdot 0}$$

$$= \int d^3p \, \frac{1}{(2\pi)^{3/2}} \cdot \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{x}} \cdot e^{-iE_p t}$$

This integral is not that simple and is used very often in Relativistic QM.

Let's do it in detail:

$$A = \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{d|p|}{(2\pi)^3} |p|^2 \cdot$$

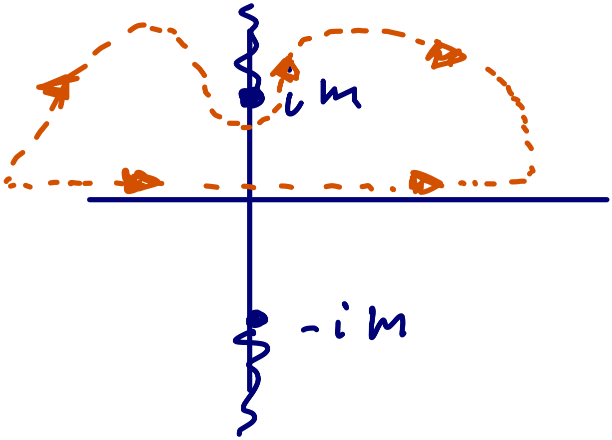
$$\int_{-1}^{1} d\cos\theta \; e^{i|p|\cdot|x|\cdot\cos\theta} \cdot e^{-iE_p t}$$

$$= \frac{1}{(2\pi)^2} \frac{1}{i|x|} \int_0^{\infty} d|p|\cdot|p| \Big(\left(e^{i|p|\cdot|x|} - e^{-i|p||x|}\right) e^{-iE_p t}$$

$$= \frac{-i}{(2\pi)^2|x|} \int_{-\infty}^{\infty} d|p| |p| e^{i|p||x|} e^{-it\sqrt{|p|^2+m^2}}$$

The complex integral

will be taken along this path



[see a handout I sent you by email for mid-term]

if we substitute $|p| = iz$

the integral becomes

$$A = \frac{-i}{(2\pi)^2 |x|} \int_m^\infty d(iz) \cdot iz \cdot e^{-z|x|}$$

$$\left(e^{t\sqrt{z^2-m^2}} - e^{-t\sqrt{z^2-m^2}} \right)$$

$$= \frac{i}{2\pi^2 |x|} e^{-m|x|} \int_m^\infty dz \cdot$$

$$\cdot z \cdot e^{-(z-m)|x|} \sinh(t\sqrt{z^2-m^2})$$

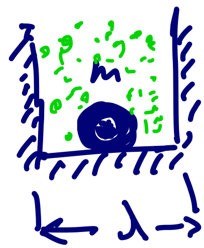
> 0 and $\neq 0$! $\sim e^{-m|x|}$

It is small but non zero and it's unacceptable as we failed to reconcile QM and relativity.

What to do?

Note it becomes even worse.

Consider a particle of mass m in the box.



a compton $\lambda = \frac{\hbar}{mc}$

or $\frac{\hbar}{m}$ if $c=1$

The uncertainty in the position $\Delta x \ll \lambda$ so $\Delta p \gg \hbar/\lambda = \frac{\cancel{\hbar}\, m}{\cancel{\hbar}}$

so now $\Delta p \gg m$ $\Rightarrow E \gg \frac{\Delta p^2}{2m}$ $\frac{m^2}{2m}$ $\gg m/2$

which leads to the $e^+ e^-$ pair production

Thus on average a box never can contain a single particle!?

L7 (take 2)

Advanced Quantum Mechanics

MAKING THINGS RELATIVISTIC.

Background information:

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi \qquad H = \frac{p^2}{2m} + V(r)$$

non-relativistic version.

In special relativity we can introduce 4-vector

$$p^\mu \cdot p_\mu = m^2c^2 = g_{\mu\nu}p^\mu p^\nu = [\, E/c \quad p^1 \quad p^2 \quad p^3 \,]$$

$$\begin{bmatrix} E/c \\ -p_1 \\ -p^2 \\ -p^3 \end{bmatrix} \rightarrow \frac{E^2}{c^2} = p^2 + m^2c^2$$

Lets change dynamical variables to operators $E \rightarrow H \quad p_i \rightarrow i\hbar\frac{\partial}{\partial x_i}$

$$H = \sqrt{-\hbar^2c^2\partial_\mu\partial_\mu + m^2c^4}$$

bad news we need a linear theory

Spin type and personal equ:

π-meson, Higgs boson = scalars $S=0$

Klein-Gordon eqn.

$S = 1/2$ fermions = Dirac eqn. electrons, quarks leptons = spinors

$S = 1$ photons and W and Z bosons.

= vector particles = Schrödinger eqn

discovered by A. Proca.

Deducing scalar eqn. = Klein-Gordon.

$$\left(i\hbar\frac{\partial}{\partial t}\right)\left(i\hbar\frac{\partial}{\partial t}\right)\Psi = H^2\Psi = (p^2c^2 + m^2c^4)\Psi$$

$$-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}\Psi = \left(-\hbar^2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + m^2c^2\right)\Psi \rightarrow$$

$$-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_0}\Psi = -\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + \underbrace{\frac{m^2c^2}{\hbar^2}}_{=\mu^2}\right)\Psi$$

From this we write down the famous Klein-Gordon equation:

$$\left\{\frac{\partial}{\partial x^\mu}\cdot\frac{\partial}{\partial x_\mu}+\mu^2\right\}\psi=0 \quad \text{or } [\partial_\mu\partial^\mu+\mu^2]\psi=0$$

$$\mu^2=\frac{m^2c^2}{\hbar^2} \quad (\text{units of mass})$$

$\partial_\mu\partial^\mu=\partial^\mu\partial_\mu\equiv\Box$ = d'Alamberian operator and in 4D Minkowski $\equiv$ 3D laplacian operator $\partial_i\partial_i=\partial^i\partial^i$

This equation specifically describes scalars. $S=0$ particles.

THE SOLUTION OF KLEIN-GORDON EQN.

The solution set for the KG equation can be written as

$$\psi(x)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{2VE_n/\hbar}}\Big(A_n e^{-i/\hbar(E_n t-p_n x)}+$$

$$\underbrace{+B_n^\dagger e^{i/\hbar(E_n t-p_n x)}}_{\text{absent in NRQM}}\Big)$$

The discrete set is typical for the system confined into an arbitrary box V

To rewrite the solution in terms of particles: $k_i=2\pi/\lambda_i$ and $p^i=\hbar k^i$ then:

$$p_\mu=\begin{bmatrix}E/c\\ p_i\end{bmatrix}=\begin{bmatrix}E/c\\ -p^i\end{bmatrix}=\hbar k_\mu=\begin{bmatrix}\hbar\omega/c\\ -\hbar k_i\end{bmatrix}\rightarrow$$

$$p_\mu=\begin{bmatrix}E\\ -p^i\end{bmatrix}=k_\mu=\begin{bmatrix}\omega\\ -k^i\end{bmatrix}$$

and if we recall the notation for

$$px=p_\mu x^\mu=Et-p^i x^i=Et-\bar{p}\cdot\bar{x}=p^\mu x_\mu$$

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in Natural units - $E \sim \omega$ $p_i = k_i$ & $p_\mu = k_\mu$
$px = kx$

We can now rewrite our solution $\psi(x)$ in terms of natural units; and in doing this we can turn off the dummy index n, which simply represents an individual wave in the summation.

For a free particle for a given $p = k$ we can label the mode in terms of E_k and ω_k.

$$\psi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left(A_k e^{-ikx} + B_k^+ e^{ikx} \right)$$

Definition of eigen solution:
Each eigen state has a form:

$$\psi_{k,A} = \frac{e^{-ikx}}{\sqrt{V}} \qquad \psi_{k,B^+} = \frac{e^{ikx}}{\sqrt{V}}$$

$$\int \psi_{k,A}^+ \psi_{k,A} \, dV = \frac{1}{V} \int e^{ikx} e^{-ikx} d^3x \equiv 1 \quad \text{or}$$

more generally:

$$\int \psi_{k',A}^+ \psi_{k,A} \, dV = \delta_{kk'}$$, the same for ψ_{k,B^+}

Probability Density of KG eqn

Lets 1st recall whats the probability density for Sch. eqn. (NRQM)
if we can introduce ψ, then we can calculate $\rho = \psi^* \psi$
Recall the continuity eqn. $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \rightarrow$
implies $\int_V \rho \, dV =$ CONSTANT IN TIME

Detour.

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ρ in NRQM:

$$\psi^{+} \times \Big| \left\{ \frac{\partial}{\partial t} \psi = \frac{1}{i\hbar} \left(-\frac{\hbar^2}{2M} \nabla^2 + V \right) \right\} \psi \Big\}$$

+ For a conjugat Schr. equ.

$$\left\{ \frac{\partial}{\partial t} \psi^{+} = \frac{-1}{i\hbar} \left(-\frac{\hbar^2}{2M} \nabla^2 + \underbrace{V^{+}}_{=V} \right) \psi^{+} \right\} \psi$$

$$\psi^{+} \frac{\partial \psi}{\partial t} + \frac{\partial \psi^{+}}{\partial t} \psi = \psi^{+} \frac{1}{i\hbar} \left(-\frac{\hbar^2}{2M} \nabla^2 + V \right) \psi + \left(-\frac{1}{i\hbar} \left(\left(-\frac{\hbar^2}{2M} \nabla^2 \psi^{+} + V^{+} \psi^{+} \right) \right) \right) \psi$$

or

$$\frac{\partial (\psi^{+} \psi)}{\partial t} = -\frac{\hbar}{2iM} \underbrace{\left(\psi^{+} (\nabla^2 \psi) - (\nabla^2 \psi^{+}) \psi \right)}_{\nabla \cdot [\psi^{+} (\nabla \psi) - (\nabla \psi^{+}) \psi]} +$$

$$+ \underbrace{\frac{\psi^{+} V \psi}{i\hbar} - \frac{V \psi^{+} \psi}{i\hbar}}_{=0 \quad V = V^{+}}$$

lets call $\rho \equiv \psi^{+} \psi$

and $j = \frac{\hbar}{2iM} \{ \psi^{+} (\nabla \psi) - (\nabla \psi^{+}) \psi \} \Rightarrow$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

For KG equation:

$$\left\{ \frac{\partial^2}{\partial t^2} \psi = (\nabla^2 - \mu^2) \psi \right\} \psi^{+}$$

$$- \left\{ \frac{\partial^2}{\partial t^2} \psi^{+} = (\nabla^2 - \mu^2) \psi^{+} \right\} \psi \quad \Rightarrow \quad \cancel{\mu^2 \psi^{+} \psi - \mu^2 \psi \psi^{+}} = 0$$

$$\text{LHS:} \quad \frac{\partial^2 \psi}{\partial t^2} \psi^{+} - \frac{\partial^2 \psi^{+}}{\partial t^2} \cdot \psi + \underbrace{\frac{\partial \psi}{\partial t} \frac{\partial \psi^{+}}{\partial t} - \frac{\partial \psi^{+}}{\partial t} \frac{\partial \psi}{\partial t}}_{=0} =$$

$$= \frac{\partial}{\partial t} \Big(\underbrace{\frac{\partial \psi}{\partial t} \psi^{+} - \frac{\partial \psi^{+}}{\partial t} \psi}_{\text{new LHS}} \Big)$$

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RHS before:

$$(\nabla^2\psi)\psi^\dagger - (\nabla^2\psi^\dagger)\psi + \underbrace{\nabla\psi\cdot\nabla\psi^\dagger - \nabla\psi^\dagger\cdot\nabla\psi}_{=0} =$$

$$= \underbrace{\nabla\cdot\left(\left(\nabla\psi\right)\psi^\dagger - \left(\nabla\psi^\dagger\right)\psi\right)}_{\text{new RHS}}$$

So we get:

$$i \,\Big|\, \frac{\partial}{\partial t}\underbrace{\left(\frac{\partial\psi}{\partial t}\psi^\dagger - \frac{\partial\psi^\dagger}{\partial t}\psi\right)}_{\rho\ =\ \text{probability density}} = \nabla\cdot\underbrace{\left(\left(\nabla\psi\right)\psi^\dagger - \left(\nabla\psi^\dagger\right)\psi\right)}_{j\ \text{probability current}} \rightarrow$$

$$\begin{cases} \rho^{RQ} = j^{0} = i\left(\frac{\partial\psi}{\partial t}\psi^\dagger - \frac{\partial\psi^\dagger}{\partial t}\psi\right) \text{ and} \\ \vec{j}^{RQ} = -i\left((\nabla\psi)\psi^\dagger - (\nabla\psi^\dagger)\psi\right) \end{cases}$$

Recall for NRQM:

$$\left\{ \rho^{NR} = \psi^*\psi \quad \text{and} \quad \vec{j}^{NR} = \frac{\hbar}{2Mi}\left\{\psi^\dagger(\nabla\psi) - (\nabla\psi^\dagger)\psi\right\} \right.$$

very strange NRQM and RQM are very different!

Lets introduce 4D notation:

$$j^\mu = \begin{Bmatrix}\rho \\ j\end{Bmatrix} = \begin{Bmatrix} j^0 \\ j^i \end{Bmatrix} \Rightarrow \boxed{\frac{\partial j^\mu}{\partial x^\mu} = \partial_\mu j^\mu = 0}$$

In 4D notation: the 4 divergence of the 4-current of any conserved quantity (total probability) is zero.

What about probability of K G. eq.?

For simplicity lets assume that in the $\psi(x)$ we have only A_k terms and no B_k^+

$$\rho = J^0 = i\left(\frac{\partial\psi}{\partial t}\psi^+ - \frac{\partial\psi^+}{\partial t}\psi\right) =$$

$$= \left(\sum_k \frac{\omega_k A_k}{\sqrt{2\omega_k}}\frac{e^{ikx}}{\sqrt{V}}\right)\left(\sum_{k'}\frac{A_{k'}^+ e^{ik'x}}{\sqrt{2\omega_{k'}}\sqrt{V}}\right) +$$

$$+ \left(\sum_{k'}\frac{\omega_{k'}A_{k'}^+ e^{ik'x}}{\sqrt{2\omega_{k'}}\cdot\sqrt{V}}\right)\left(\sum_k \frac{A_k}{\sqrt{2\omega_k}}\frac{e^{-ikx}}{\sqrt{V}}\right)$$

if we integrate $\int \rho\, dv = 1$

$$\frac{1}{2V}\Big(\quad\Big)\Big(\quad\Big) + \Big(\quad\Big)\Big(\quad\Big)$$

unless $k \Rightarrow k'$ $\omega_{k'} \to \omega_k$ the integral $= 0$

so we have in RQM:

RQM: $\int \underbrace{\psi^+\psi}_{\neq \rho^{RQM}}^{\rho^{NRQ}} dv = \sum_k \frac{(A_k)^2}{2\omega_k} \neq 1$ but

$$\int \underbrace{i\left(\frac{\partial\psi}{\partial t}\psi^+ - \frac{\partial\psi^+}{\partial t}\psi\right)}_{\rho} dv = \sum |A_k|^2 = 1$$

The total probability is scalar $\Rightarrow$ must be an invariant. So ρ^{RQM} is good.

Now here is the problem.

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Negative probabilities in RQM

Prove that: if a particle ψ contains only eigenstates in the form

$$e^{+i(E_n t - \bar{p}_n \cdot \bar{x})/\hbar} = e^{ikx} \xrightarrow{\text{then}}$$ (those are terms with B_k^+)

the total probability is -1.

So the extra states in RQM have physically negative probabilities!

$$\rho = j^0 = i\left(\frac{\partial \psi}{\partial t}\psi^+ - \frac{\partial \psi^+}{\partial t}\psi\right)$$

lets try the simplest solution:

$$\psi = A e^{-ipx} \qquad \psi^+ = A^+ e^{ipx} \qquad px = -iEt + i\bar{p}\bar{x}$$

$$= i\left(A(-iE)e^{-ipx} \cdot A^+ e^{+ipx}\right) -$$

$$- i\left(A A^+ (+i) e^{+ipx} e^{-ipx}\right) =$$

$$= 2|A|^2 E$$ E can be positive or negative

this in turn means that

ρ can be negative probability density.

This may be very bad, but wait lets see what Feynman and Stueckelberg can tell us.

Interpretation of negative energy states.

N.E.S are particles moving back in time.
$\equiv$ antipartcles

Consider classical electrodynamics:

$$m \frac{d^2 x^\mu}{d\tau^2} = q F^\mu_{\ \nu} \frac{dx^\nu}{d\tau}$$

charge EM tensor

notice $\tau \to -\tau$ is the same as $q \to -q$.

particle moving backword in time =
= antiparticle moving forward in time

Based on this one way to eliminate all the negative states is turn their momentum and charges into opposite and thus generate antiparticles instead. , i.e

$E t - p \cdot x$ $t \to -t$ means $E \to -E$

but we need to reverse $p \to -p$

① $E(-t) - p \cdot x$
② $-(Et + px)$
③ $-(Et - px)$

Lets examine the EM current density for KG equation.

$$j^\mu_{em} = q j^\mu$$

where $j^\mu = i\{\psi^+ \underset{\partial^\mu}{\nabla} \psi - \underset{\partial^\mu}{\nabla} \psi^+ \psi\}$ ← as a 4-vector

for $\psi = N e^{-ipx}$ $\Rightarrow$ $j^\mu_{em} = q \cdot 2|N|^2 p^\mu = q \cdot 2|N|^2 (E, \bar{p})$

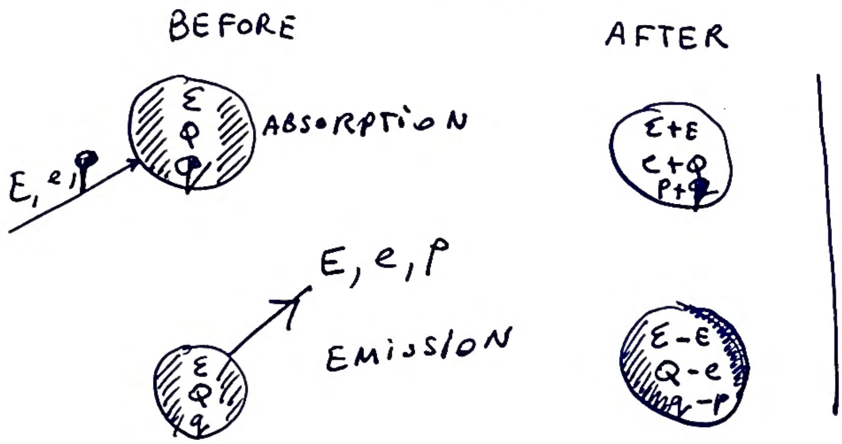
For negative E states $j^\mu_{em} = -q \cdot 2|N|^2 (E, \bar{p})$

Recall negative states are bad since they mess up the probabilities, so we want to turn these states into $E>0$:

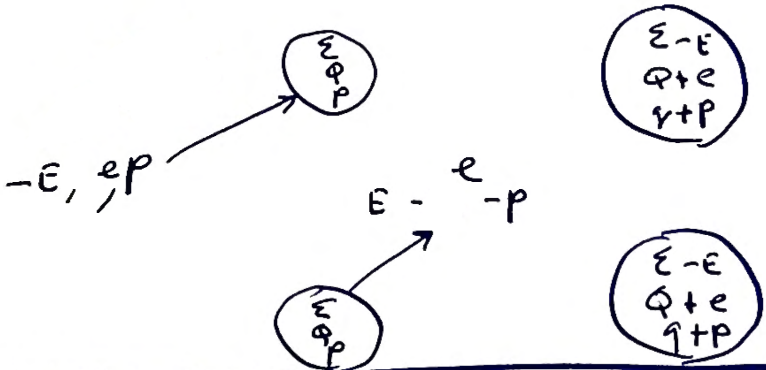
$$\begin{matrix} q \to -q \\ p \to -p \end{matrix} \Rightarrow j^{\mu}_{em} = (-q)\, 2|N|^2 (E, -p)$$

or incoming particle is outgoing antiparticles.

Q: Can we apply this protocol to the <u>interacting particles</u>. The answer is yes.



⇒ this is normal



absorption of negative E particles = emission of negative and negative e but <u>positive E</u>

Rule:

$E = \pm\sqrt{p^2 + m^2c^4}$ but we accept only (+)

but negative E are antiparticles

$$\Psi(x) = \underset{e^{-i(Et - p\cdot x)}}{\text{incoming } +E} + \underset{e^{+(Et - px)}}{\text{outgoing } +E \text{ antiparticles}}$$