HOMEWORK # 3 SOLUTION

1. $\hat{H}_{\omega} = \bar{h} \cdot \bar{\sigma} = \epsilon_0 [\sigma_x \sin\theta \cos\varphi + \sigma_y \sin\theta \sin\varphi + \sigma_z . \cos\theta]$

b/c $h = \epsilon_0 \hat{r}$, $x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

$\vec{\omega} = (\theta, \varphi)$ is the 2-component vector of parameters for this problem

2. The 2 eigenstates for the hamiltonian

$$\epsilon_{\omega}^{(1)} = -\epsilon_0 \quad \text{and} \quad \epsilon_{\omega}^{(2)} = \epsilon_0$$

w/ the corresponding eigenvectors

$$|\varphi_{\omega}^{(1)}\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{i\varphi} \\ -\cos\theta/2 \end{pmatrix} \quad |\varphi_{\omega}^{(2)}\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix}$$

Can be checked by plug.in those into the

$$h|\psi\rangle = \epsilon|\psi\rangle$$

3. The eigenfunctions are orthogonal

i.e. $\langle \psi_{\omega}^{(1)} | \psi_{\omega}^{(2)} \rangle = \left(\sin\theta/2 \, . e^{i\varphi} \quad -\cos\theta/2\right)^{*} .$

$$\cdot \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix} = 0$$

and they are properly normalized

$$\langle \psi_w^{(1)} | \psi_w^{(1)} \rangle = \langle \psi_w^{(2)} | \psi_w^{(2)} \rangle =$$

$$= \sin^2 \theta/2 + \cos^2 \theta/2 = 1$$

4. The spin at North pole $\theta = 0$

$$|\psi_0^{(1)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{and} \quad |\psi_0^{(2)} \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and for the south pole $\theta = \pi$

$$|\psi_\pi^{(1)} \rangle = \begin{pmatrix} e^{-i\varphi} \\ 0 \end{pmatrix} \qquad |\psi_\pi^{(2)} \rangle = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix}$$

But these are not well defined b/c φ can have any value in the $[0, 2\pi]$ for the same position of the spin; in other words the w.f. is multiply defined

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$$A_\theta^{(1)}(\theta, \varphi) = i \langle \psi_w^{(1)} | \frac{\partial}{\partial \theta} | \psi_w^{(1)} \rangle$$

$$= i \begin{pmatrix} \sin\theta/2 \, e^{-i\varphi} \\ -\cos\theta/2 \end{pmatrix}^+ \cdot \begin{pmatrix} \frac{1}{2} \cos\theta/2 \, e^{-i\varphi} \\ 1/2 \sin\theta/2 \end{pmatrix} = 0$$

$$A_\varphi^{(1)}(\theta, \varphi) = i \langle \psi_w^{(1)} | \frac{\partial}{\partial \varphi} | \psi_w^{(1)} \rangle = i \begin{pmatrix} \sin\theta/2 \, e^{-i\varphi} \\ -\cos\theta/2 \end{pmatrix}^+ \cdot \begin{pmatrix} -i \sin\theta/2 \, e^{-i\varphi} \\ 0 \end{pmatrix} = \frac{1}{2} - \frac{1}{2}\cos\theta$$

6. Berry's curvature:

$$\underline{\Omega^{(1)}_{\theta\varphi}(\theta,\varphi)} = \underbrace{\frac{\partial A^{(1)}_{\varphi}}{\partial\theta}} - \frac{\partial A^{(1)}}{\partial\varphi} =$$

$\hookrightarrow = 0$ from 5

$$= \underline{\frac{1}{2}\sin\theta}$$

7. $$\underline{\gamma^{(1)}(\theta)} = \oint_C A^{(1)}(w')\,dw' = \int_0^{2\pi} A^{(1)}_{\varphi}(\theta,\varphi')\,d\varphi'$$

$\uparrow$ $\frac{1-\cos\theta}{2}$

$$\underline{= \pi(1-\cos\theta)}$$

here we integrate over the closed path around the horizontal cirle C, see figure in the handout for HW3.
with $\overline{w}' = (\theta,\varphi')$ so the path is described by $dw' = \varphi\, d\varphi$ with $\varphi = [0,2\pi)$

8. The Berry's curvate is integrated over enclosed by the circle C.

$$\underline{\gamma^{(1)}_{\theta}} = \int_0^{2\pi} d\varphi' \int_0^{\theta} \Omega^{(1)}_{\theta\varphi}(\theta'\varphi')\,d\theta'$$

$$= 2\pi \int_0^{\theta} \frac{1}{2}\sin\theta\, d\theta = \underline{\pi(1-\cos\theta)}$$

if we let $\theta \to \pi$ $\lim_{\theta\to\pi} [\pi(1-\cos\theta)] = 2\pi$

9. Chern number

$$= \frac{\gamma}{2\pi} = 1$$