# Lecture 8

## Spin of Dirac Particles

Lets discuss spin in more detail.
In the Heisenberg picture we get

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

Back to our original notation:
$\gamma_\mu = (\beta, -i\beta\alpha_\mu)$

For a free Dirac particle $H = c\cdot\alpha\cdot p + \beta mc^2$
Lets introduce a time evolution for this particle with an angular momentum $\hat{L}$:

$$-i\hbar\frac{\partial}{\partial t}\vec{L} = [H, \vec{L}]$$

$\vec{L} \to (L_x, L_y, L_z)$

Consider only one component; say $L_3$

$$-i\hbar\frac{\partial L_3}{\partial t} = [H, L_3] = [\sum_{k=1}^{3} c\,\alpha_k p_k + \beta mc^2, \underbrace{x_1 p_2 - x_2 p_1}_{=L_3}]$$

recal $\alpha_k$ and $\beta$ are matrices and

$[\beta mc^2, x_1 p_2 - x_2 p_1] = 0 \leftarrow \beta$ is a matrix, $x$ and $p$ are scalars

$$-i\hbar\frac{\partial L_3}{\partial t} = [\sum c\,\alpha_k p_k, x_1 p_2] - [\sum c\,\alpha_k p_k, x_2 p_1] = -i\hbar c\sum \alpha_k \delta_{k1} p_2 + i\hbar c\sum \alpha_k \delta_{k2} p_1$$

$= -i\hbar c(\alpha_1 p_2 - \alpha_2 p_1) = -i\hbar(\alpha \times p)_3$ 3rd component

Here I used $[p_i, x_j] = -i\hbar\delta_{ij}$ $[p_i, p_j] = 0$

the same we can derive for $L_2$ and $L_1$

in other words:

$$\frac{d\vec{L}}{dt} = c(\alpha \times p) \neq 0$$

thus $\vec{L}$ is not a constant of motion.

We should construct a new operator to cancel out $-c(\alpha \times p)$ and make it a good operator. i.e,

$\vec{L} + \vec{A} =$ is conserved if $\frac{d\vec{A}}{dt} = -c(\alpha \times p)$

We now can demonstrate that

$$\vec{A} = \frac{\hbar}{2} \cdot \vec{\sigma}^* \quad \text{where} \quad \sigma^* = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

For this purpose we calculate

$$-i\hbar \frac{d\sigma_3^*}{dt} = [H, \sigma_3^*] =$$

$= [\sum c\alpha_k \cdot p_k + \beta mc^2, \sigma_3^*] = 2i(\alpha_1 p_2 - p_1 \alpha_2)$

Show This!

again we use $[\beta mc^2, \sigma_3^*] = 0$

$[\alpha_3, \sigma_3^*] = 0$

$\sigma_1^* \sigma_3^* = -\sigma_3^* \sigma_1^*$

and $\sigma_1^* \sigma_3^* = -i\sigma_2^*$ (I used sympy) that is : $\frac{\hbar}{2}\frac{d\sigma_3^*}{dt} = - c\,(\alpha \times p)_3 \leftarrow 3^d$ component

the same for $\sigma_1^*$ and $\sigma_2^*$, in short

$$\frac{d}{dt}\frac{\hbar}{2}\vec{\sigma}^* = - c\,(\alpha \times \bar{p})$$

Combining this with the expression for $\vec{L}$

$$\frac{d\bar{L}}{dt} = c(\alpha \times \bar{p}) \Rightarrow \frac{d}{dt}\Big(\bar{L} + \underbrace{\frac{\hbar}{2}\bar{\sigma}^*}_{\text{spin angular moment}}\Big) = 0$$

Thus a new quantity:

$$\boxed{\bar{J} = \bar{L} + \frac{\hbar}{2}\bar{\sigma}^*}$$

is conserved $\frac{d\bar{J}}{dt} = 0$

and the Dirac particle has spin $\frac{\hbar}{2}\sigma^*$

## Dirac PARTICLE IN A 1D POTENTIAL

To solve a problem of a Dirac particle in the potential $V$ we 1st modify the equation to include the potential.

Intuitively we can write down:

$$H = c\alpha \cdot \bar{p} + \beta m c^2 + V(z) =$$

$$= -i\hbar c \frac{d}{dz}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc^2 + V(z)$$

where we selected $\alpha \equiv \alpha_3$

for $H\psi = E\psi \Rightarrow$

$$\left[ i\hbar c \frac{\partial}{\partial z}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + mc^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V \right]\begin{pmatrix} v \\ w \end{pmatrix} =$$

$$= E\begin{pmatrix} v \\ w \end{pmatrix}$$ $\Rightarrow$ multiplying those matrices

$$\begin{cases} -i\hbar c \underbrace{w'}_{\frac{dw}{dz}} + v(mc^2 + V) = Ev \\ \hbar c v' - w(mc^2 + V) = Ew \end{cases}$$

Recall $\overset{\text{for } E_1}{\Psi_1} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$ and $\overset{\text{for } E_2}{\Psi_2} = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$

those are two bound state Solutions

So we have 4 eqns for those components

$$(-)\, v_2^* \quad -\hbar c\, w_1' + (mc^2+V)\, v_1 = E_1 w_1$$
$$w_2^* \quad \hbar c\, v_1' - (mc^2+V)\, w_1 = E_1 v_1$$
$$(-)\, w_1^* \quad -\hbar c\, w_2' + (mc^2+V)\, v_2 = E_2 v_2$$
$$v_1^* \quad \hbar c\, v_2' - (mc^2+V)\, w_2 = E_2 w_2$$

$$\begin{cases} \hbar c\,(v_1 w_2' - w_1' v_2) = (E_1 - E_2)\, v_1 v_2 \\ \hbar c\,(v_1' w_2 - w_1 v_2') = (E_1 - E_2)\, w_1 w_2 \end{cases}$$

or

$$\hbar c \frac{d}{dz}\left(v_1 w_2 - w_1 v_2\right) = (E_1 - E_2)\, \psi_1^* \psi_2$$

① if $E_1 = E_2 \Rightarrow \dfrac{d}{dz}\left(v_1 W_2 - v_2 w_1\right) = 0$

$\downarrow$ const.

Degeneracy investigated

We assume that $v$ and $w \to 0$ if $z \to \infty$

so the constant $= 0$

that is: $\dfrac{v_1}{w_1} = \dfrac{v_2}{w_2}$, Returnin

or $\dfrac{v_1}{v_2} = \dfrac{w_1}{w_2} \Rightarrow \psi_1 \propto \psi_2$

meaning they represent the same state

and meaning also they are NON-DEGENERATE

and also $\underbrace{\langle \psi_1 | \psi_2 \rangle}_{=0} = \int_{-\infty}^{\infty} \psi_1^* \psi_2\, dz \Longrightarrow \longrightarrow$

recall $\hbar c \frac{d}{dz}(v_1 w_2 - w_1 v_2) = (E_1 - E_2)\psi_1^* \psi_2$

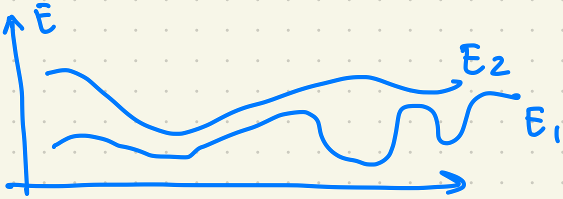
$$\psi_1 \psi_2^* = \underbrace{\frac{\hbar c}{E_1 - E_2} \frac{d}{dz}(v_1 w_2 - w_1 v_2)}_{} \Rightarrow \int \ldots$$

$$\frac{1}{E_1 - E_2}(v_1 w_2 - w_1 v_2)\Big|_{-\infty}^{\infty} = 0$$

or $\psi_1$ and $\psi_2$ are orthogonal

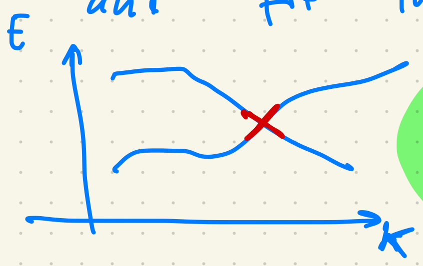
+ non-deneracy
This means that level crossing cannot occur!

When the potential varies smoothly the w.f. also vary smoothly




x ← some parameter e.g. k momentum

and for the levels to cross

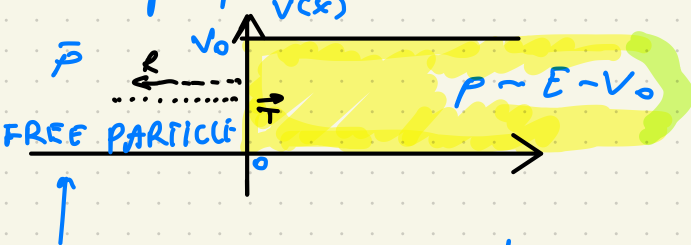



this requiare a SINGULAR behavior in $\langle \psi_1 | \psi_2 \rangle$ when $E_1 = E_2$

AND SINGULAR POTENTIAL!

# Klein "Paradox"
(verified in graphene)

Consider a scattering process by the step potential




$1^{st}$ we construct the plane wave solution of Dirac's free electron in 1D:

$$E\psi = (c\,\alpha_x p_x + \beta mc^2)\psi \quad \text{or}$$

$$i\hbar \frac{\partial \psi}{\partial t} + i\hbar c \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \frac{\partial \psi}{\partial x} - mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \psi = 0$$

or in the $x$-component form:

$$i\hbar \frac{\partial \psi_1}{\partial t} + i\hbar c \frac{\partial \psi_4}{\partial x} - mc^2 \psi_1 = 0$$

$$i\hbar \frac{\partial \psi_2}{\partial t} + i\hbar c \frac{\partial \psi_3}{\partial x} - mc^2 \psi_2$$

$$i\hbar \frac{\partial \psi_3}{\partial t} + i\hbar c \frac{\partial \psi_2}{\partial x} - mc^2 \psi_3 = 0$$

$$i\hbar \frac{\partial \psi_4}{\partial t} + i\hbar c \frac{\partial \psi_1}{\partial x} - mc^2 \psi_4 =$$

Note: $\psi_1$ couples only to $\psi_3$ and $\psi_2$ to $\psi_4$.

b/c of this let's into a 2-component ⑧
spinor, with $\psi_u = \psi_1$ or $\psi_2$
$\psi_\ell = \psi_3$ or $\psi_4$

for stationary states we get:

$$\begin{cases} i\hbar c\,\psi_1' + (E + mc^2)\psi_4 = 0 & i\hbar c\,\psi_2' + (E+mc^2)\psi_3 = 0 \\ i\hbar c\,\psi_3' + (E - mc^2)\psi_2 = 0 & i\hbar c\,\psi_4' + (E - mc^2)\psi_1 = 0 \end{cases}$$

in terms of $\psi_\ell$ and $\psi_u$ we can rewrite it as a single equation

$$\begin{cases} i\hbar c\,\psi_u' + (E + mc^2)\psi_\ell = 0 \\ i\hbar c\,\psi_\ell' + (E - mc^2)\psi_\ell = 0 \end{cases}$$

rename $\psi_u = u$ and $\psi_\ell = w$

$$\begin{cases} i\hbar c\,v' + (E + mc^2)\,w = 0 & \text{V} \\ i\hbar\,c w' + (E - mc^2)\,v = 0 & \text{VV} \end{cases}$$

$\frac{d}{dx}$ V and using $w'$ from VV

$$i\hbar c\,v'' + (E + mc^2)\,w' = 0$$

$$w' = \frac{v\,(E - mc^2)}{i\hbar}$$

$$v'' + \frac{p^2}{\hbar^2}v = 0 \quad \text{where } p^2 \equiv \frac{1}{c^2}(E^2 - m^2c^4)$$

and putting $v''$ into ● $\quad w = -\frac{i\hbar c}{E + mc^2}\cdot v'$

For $v'' + \frac{p^2}{\hbar^2} v = 0$

- $v = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$
- and $w = \frac{-(i\hbar c)}{E + mc^2} \cdot v'$

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Based on this we can write down:

$$v_<(x) = A\left(e^{ipx/\hbar} + R\,e^{-ipx/\hbar}\right)$$

$$w_<(x) = A\left(a\,e^{ipx} - aR\,e^{ipx/\hbar}\right)$$

$a \equiv cp/(E + mc^2)$

From solution for $<$ we get the solution for $>$ by Replacing $E$ by $E - V_0$

The w.f. for $x<0$ is:

$$\psi_< = \begin{pmatrix} v_< \\ u_< \end{pmatrix} = A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} e^{ipx/\hbar} + R\begin{pmatrix} 1 \\ -a \end{pmatrix} e^{ipx/\hbar}\right] \equiv A\left[u_+ e^{ipx/\hbar} + R\,u_- e^{-ipx/\hbar}\right]$$

where $u_\pm = \begin{pmatrix} 1 \\ \pm a \end{pmatrix}$

for $x>0$ there is no reflected wave

so $\psi_> = \begin{pmatrix} v_> \\ w_> \end{pmatrix} = D\,\bar{u}\,e^{ipx/\hbar}$

here $\bar{u} = \begin{pmatrix} 1 \\ b \end{pmatrix}$ $\bar{p} = p(E - V_0) = \frac{1}{c}[(E-V_0)^2 - m^2c^4]^{1/2}$

and $b = \dfrac{c\bar{p}}{(E - V_0 + mc^2)}$

As usual to determine those constants A, D and R we use

$$\begin{cases} \psi_<(0) = \psi_>(0) \\ \psi'_<(0) = \psi'_>(0) \end{cases}$$

for $\psi$: $D\bar{u}\, \cancel{e^{ipx/\hbar}} = A[u_+ \cancel{e^{ipx/\hbar}} + Ru_- \cancel{e^{-ipx/\hbar}}]$ $\downarrow_{x=0}$

$\psi'$: . . . . .

$A(U_+ + RU_-) = D\bar{u} \Rightarrow$

$$A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} + R\begin{pmatrix} 1 \\ -a \end{pmatrix}\right] = D\begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\Rightarrow \begin{cases} A(1+R) = D \\ Aa(1-R) = bD \end{cases} \Rightarrow R = \frac{a-b}{b+a}$$

$$T = \frac{2a}{a+b}$$

Behavior of the w. f. depends on $V_0$ 

Consider ① $E > V_0 + mc^2$

② $V_0 - mc^2 < E < V_0 + mc^2$

③ $E < V_0 - mc^2$

① $E > V_0 + mc^2 \Rightarrow$

$$E^2 > m^2c^4 + \underset{>0}{V_0^2} + \underset{>0}{2mc^2 V_0} \Rightarrow$$

$E^2 - m^2c^4 > 0$

$p^2c^2 = E^2 - m^2c^4$ also $> 0 \Rightarrow$

$\underline{p \text{ is real}}$

Since $E - V_0 > mc^2$ $(E - V_0)^2 >$
$> m^2c^4$ or

$$\bar{p} = \left[(E - V_0)^2 - m^2c^4\right]^{1/2} > 0$$

$\Rightarrow$ and $\bar{p}$ $\underline{\text{is real}}$

as such if

$\boxed{E > V_0 + mc^2}$ ALL LIKE NRQM!

$x < 0$: $\underset{\text{incoming}}{e^{ipx}} + \underset{\text{reflected}}{e^{-ipx}}$ $x > 0$ $\underset{\text{transmitted}}{e^{ipx}}$

2 $V_0 - mc^2 < E < V_0 + mc^2$

in this case $(E - V_0)^2 < m^2c^4$

or $\bar{p} = \left[ (E - V_0)^2 - m^2c^4 \right]^{1/2}$

is IMAGINARY

Then we have for:

$x < 0$ $x > 0$

incoming + reflected exponentially decaying wave

LIKE IN NRQM.

Finally

3 $E < V_0 - mc^2$

$E - V_0 < -mc^2 \Rightarrow$

$(E - V_0) < 0$ and

$(E - V_0)^2 > m^2c^4 \Rightarrow$

$\bar{p}$ is Real !

Meaning that we have oscillatory behaviour after the barrier!!??

Recall for NRQM no such solution is possible.

But wait as $E_0 - V_0 + mc^2 < 0$

and thus $b$ is negative

$$\left( b = \frac{c\,\overbrace{p}^{>0}}{(\underbrace{E - V_0 + mc^2}_{<0})} \Rightarrow b < 0 \right)$$

we can write down

$$\boxed{|R| = \left| \frac{a-b}{a+b} \right| > 1}$$

Klein paradox: The amplitude of the reflected wave is LARGER than incoming one.

or

More particles gets reflected than arrived

Also one can show that even for $V_0 \to +\infty$

$$T = \frac{2p}{E+p} \neq 0 \;!!\,??$$

# Dirac electron in the field

Electric field is described by
$A_\mu = \left(\frac{i\varphi}{c}, \vec{A}\right)$ and Dirac eqn
is simply is modified as $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$
$E \to E - e\phi$

$$\begin{cases} (\overset{E-e\varphi}{E} - mc^2)\, v = c\,(\sigma\cdot p)\, w \\ (\underset{E-e\varphi}{E} + mc^2)\, w = c\,(\sigma\cdot p)\, v \end{cases}$$

Now we are interested in positive solutions in the form

$$E = \underset{>0}{\mathcal{E}} + mc^2$$

$$\left(2mc^2 + \overset{0}{\mathcal{E}} - \overset{0}{e\varphi}\right) w = c\sigma\cdot\left(p - \frac{e}{c}A\right) v$$

in a weak field $\mathcal{E}$ and $e\phi$ are small

so $\quad 2mc^2 w \cong c\sigma\cdot\left(p - \frac{e}{c}A\right) v \Rightarrow$

$$w \simeq \frac{1}{2mc}\,\sigma\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) v$$

$$(\underset{E = \mathcal{E} + mc^2}{E} - e\varphi - mc^2)\, v \cong \frac{c\left(\sigma\cdot\left(p - \frac{e}{c}A\right)\right)^2}{2mc} v \Rightarrow$$

$$\frac{1}{2m}\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 v = (\mathcal{E}-e\varphi)v$$

using $(\sigma\cdot B)(\sigma\cdot C) = B\cdot C + i\sigma\cdot(B\times C)$

with $B = C = \left(p-\frac{e}{c}A\right)$ $\Rightarrow$ $i\hbar\nabla$

$$\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 = \left(p-\frac{eA}{c}\right)^2 + i\sigma\left\{\left(p-\frac{e}{c}A\right)\times\left(p-\frac{e}{c}A\right)\right\} = \left(p-\frac{eA}{c}\right)^2 - \frac{e\hbar}{c}\sigma\cdot B$$

$$p\times p + \frac{e^2}{c^2}A\times A + \underbrace{p\times A - A\times p}_{=2p\times A} \qquad p = i\hbar\nabla$$

where $\bar{B} = \bar{\nabla}\times\bar{A}$

thus the eqn:

$$\frac{1}{2m}\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 v = (\mathcal{E}-e\varphi)v$$

becomes

$$\boxed{\frac{1}{2m}\left[\left(p-\frac{e}{c}A\right)^2 - \frac{e\hbar}{2mc}\sigma\cdot B + e\varphi\right]v = \mathcal{E}v}$$

$B = \nabla\times B$

PAULI EQUATION

the extra term $-\dfrac{e\hbar\sigma\cdot B}{2mc}$ suggest that an electron in the magn. field gains extra energy $-\bar{\mu}\cdot\bar{B} = -\dfrac{e\hbar\sigma}{2mc}\cdot B = -\mu_B^{\sigma}\cdot B$

Important topic is spin-orbit interaction: READ pp 493-495 of the text.

the END OF RQM section!