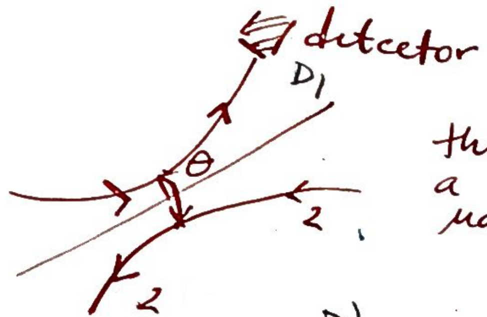
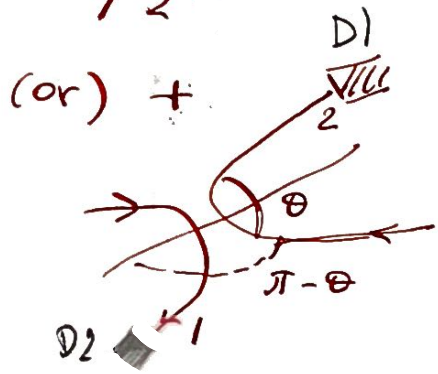
Lecture #6

Collission between identical particles

Also read Feynman's Lectures Ch. 4-1



this is a center of mass picture.



So we have two independent channels. And we cannot decide which way we scatter particles 1 and 2, and we have to sum up the amplitudes for both events.

Note, classically we would have a differential cross-section whis is the SUM of σ_i meaning we adding up probabilities ~~and~~ not amplitudes.

$$\frac{d\sigma_{cm}}{d\Omega} = |f(\theta,\phi)|^2 + |f(\pi-\theta, \phi+\pi)|^2$$

and f is defined from the scattered w.f.

$$\psi_{r\to+\infty} \to e^{ikr} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

Now recall the product for bosons and fermions is very different.

① Lets assume we scatter 2 spinless bosons

$$r_1 \to r_2 \quad \to \quad |r| \underset{r_1\to r_2}{\overset{\uparrow}{=}} r_1 - r_2 \implies r \to -r.$$

and in the polar coordinates means that $r, \theta-\pi, \phi+\pi$

2 spinless bosons

$$\Psi_s(r \to +\infty) \overset{\Downarrow}{=} e^{ikr} + e^{ikr} + [f(\theta,\phi) + f(\pi-\theta, \pi+\phi)] \frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi) + f(\pi-\theta, \phi+\pi)|^2$$

$$= |f(\theta,\pi)|^2 + |f(\theta, \phi+\pi)|^2 + \underline{2\,\mathrm{Re}\,|f(\theta,\varphi) \cdot f^*(\pi-\theta, \varphi+\pi)|}$$

this is extra compared to classical scattering. = = interference between scattering amplitudes

If the potential is independent of ϕ (e.g. central potential) $\Rightarrow$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 + 2\,\mathrm{Re}\,[f(\theta) \cdot f(\pi-\theta)]$$

Note if $\theta = \frac{\pi}{2}$ $\rightarrow$ $\frac{d\sigma}{d\Omega} = \underline{\underline{4}}\,|f(\theta)|$

$\uparrow$ the symmetry angle in the c. of m.

$\uparrow$ so $1 + 1 = 4$! in quantum mechanics

Moreover recall in the phase shift analysis Lecture /

$$f(\theta) \propto \frac{1}{k} \sum_{\ell=0}^{\ell_{max}} (2\ell+1)\, \underline{P_\ell(\cos\theta)}\, e^{i\delta_\ell} \sin\delta_\ell$$

to be symmetric $\theta \to \pi - \theta$, it can contain only even ℓs.

$(P_\ell(-x) = (-1)^\ell P_\ell(x))$

② Scatterings of 2 fermions spin = 1/2

The total wave. func. must be antisymmetric.

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The <u>spin part</u> of the w.f. can be symmetric ($\uparrow\uparrow$) or antisymm. ($\uparrow\downarrow$), then the spatial part must be symm. for $\uparrow\downarrow$ and antisymmetric for $\uparrow\uparrow$. Assume the potential is central and spin independent. $\Rightarrow$

see previous lecture

$$\left.\begin{aligned} f_s &= f(\theta) + f(\theta - \pi) \\ f_a &= f(\theta) - f(\theta - \pi) \end{aligned}\right\} \Rightarrow$$

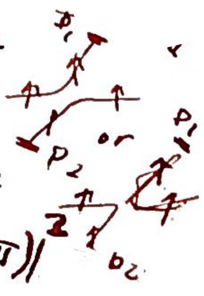
$$\left.\frac{d\sigma}{d\Omega}\right|_{\uparrow\downarrow} = |f(\theta)|^2 + |f(\pi-\theta)|^2 + 2\,\mathrm{Re}\left[f(\theta) f^*(\pi-\theta)\right]$$

and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\uparrow} = |f(\theta)|^2 + |f(\pi-\theta)|^2 - 2\,\mathrm{Re}\left[f(\theta) f^*(\pi-\theta)\right]$$

Assume that incoming fermions are up polarized. e.g.

Fraction	S1	S2	Spin in D1	Spin in D2	Probability
1/4	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\uparrow$	$\downarrow$	$\{\uparrow$ / $\downarrow$	$\{\downarrow$ / $\uparrow$	$\lvert f(\theta)\rvert^2$ / $\lvert f(\theta-\pi)\rvert^2$
1/4	$\downarrow$	$\uparrow$	$\{\uparrow$ / $\downarrow$	$\{\downarrow$ / $\uparrow$	$\lvert f(\theta-\pi)\rvert^2$ / $\lvert f(\theta)\rvert^2$



Total: $= \frac{1}{2}\left[\,|f(\theta) - f(\pi-\theta)|^2 + \frac{1}{2}|f(\theta)|^2 + \frac{1}{2}|f(\theta-\pi)|^2\right.$

Total cross section unpolarized $= \frac{1}{4}\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\downarrow} + \frac{3}{4}\left(\frac{d\sigma}{d\Omega}\right)_{\uparrow\uparrow}$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - \cancel{2}\,\cancel{\tfrac{1}{2}}\, f(\theta)\cdot f^*(\pi-\theta)|$$

Compared to bosons the cross-section is a factor of 4 less.

Also: at $\theta = \frac{\pi}{2}$ $\quad \left(\frac{d\sigma}{d\Omega}\right)^{\text{fermions}}_{\theta=\pi/2} = |f(\theta=\pi/2)|^2$

for bosons $\theta = \pi/2$ $\quad \left(\frac{d\sigma}{d\Omega}\right)^{\text{bosons}}_{\theta=\frac{\pi}{2}} = 2|f(\theta=\pi/2)|^2$

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Occupation number representation

This idea is very useful for many body theory or quantum field thery.

1. Particle in the box of size L.

lets set $\hbar = 1$ $\quad p = -i\frac{\partial}{\partial x}$; $\psi(x) = \frac{1}{\sqrt{L}} e^{ipx}$

$$p\,\psi(x) = -i\frac{\partial \psi}{\partial x} = p\,\psi(x)$$

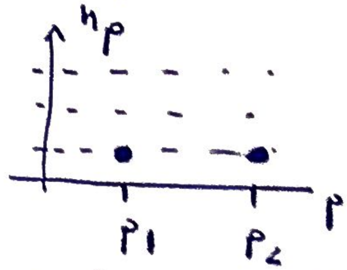
if $\psi(x) = \psi(x+L)$ $\quad e^{ipx} = e^{ip(x+L)} \Rightarrow p_m = \frac{2\pi m}{L}$

NEW NOTATION

onto a multi-particle state (e.g. bosons)

$$p\,|p_1 p_2\rangle = (p_1 + p_2)\,|p_1 p_2\rangle$$

$$H\,|p_1 p_2\rangle = (E_1 + E_2)\,|p_1 p_2\rangle$$



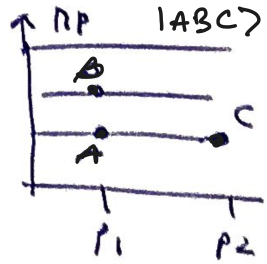
a two particle state $|p_1 p_2\rangle$

What if I have 2 particles in p_3? $\quad E_{p_3} = 2 \times E_{p_3}$ double of a single particle energy

In general:

$$\sum_m n_{p_m} E_{p_m}$$

n_{p_m} is the total number of particles in the state p_m



In QFT instead of listing what particle is in what momentum state we can say

two particles are in p_1, 1 particle in p_2 etc.

so we just specify how many in what state $p_1 \ldots p_N$

e.g. $|2100\ldots\rangle$

number of particles in this momentum state.

is called $\boxed{\text{occupation number representation}}$

e.g. $|q_1 q_1\rangle = |20\rangle$ $\quad |q_2 q_2\rangle = |02\rangle$ $\quad |q_1 q_1 q_1\rangle = |30\rangle$

$|q_1 q_2\rangle = |11\rangle$ etc.

old

new notation

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What happens when we act on this state by H

$$H|n_1 n_2 \dots\rangle = \left[\sum_m n_{p_m} E_{p_m}\right] |n_1 n_2 n_3 \dots\rangle$$

simply we find out how many particles in that state × Energy of that state

Big Q: Why do we care?

Recall in harmonic oscillator

$E_n = (n+\frac{1}{2})\hbar\omega$ or $E_n = n\hbar\omega$ ← $\left\{ \hbar\omega \frac{1}{2} = \text{just newly redefined "0"} \right\}$

so in the oscillat we have n quanta. and the energy between states is equally spaced.

Now imagine N independent oscillators each labeled by k and the spacing is $\hbar\omega_k$

The total $E = \sum_{k=1}^{N} \hbar\omega_k \cdot n_k$ ← so the k^{th} oscillator has n_k quanta in it.

e.g $k=3$ $\hbar\omega_3$ oscillator has n_3 quanta in it and contributes to the energy $\hbar\omega_3 \cdot n_3$

In general

$$E = \sum_m n_{p_m} E_{p_m}$$

, so we say,

the momentum state p_m has n_{p_m} particles in it and contributes $n_{p_m} E_{p_m}$ energy.

So it looks like we can think of a general system as analogous to oscillators.

SUMMARY:

Quanta in oscillators	$\longrightarrow$	Particles in momentum states
kth oscillator	$\longrightarrow$	mth momentum mode p_m
$E = \sum_{k=1}^{N} n_k \hbar\omega_k$	$\longrightarrow$	$E = \sum_{m=1}^{N} n_{p_m} E_{p_m}$.

VERY IMPORTANT STEP

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→ REPLACE STATE VECTOR WITH AN OPERATOR!

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What's next: Can we remove the the notion of state vectors at all?

$$|n_1 n_2 \dots\rangle = \prod_k \frac{1}{(n_k!)^{1/2}} (a_k^\dagger)^{n_k} |0\rangle$$

A very special state $|0\rangle$

so we retain only one very special state $|0\rangle$ VACUUM

From $\Big\{ |n_1 n_2 \dots n_N\rangle = \frac{1}{\sqrt{n_1! n_2! \dots n_N!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_n^\dagger)^{n_N} |0,0,0\dots\rangle \Big\}$

The general state of the harmonic oscillator:

$$|n_1 \dots n_N\rangle = \prod_k \frac{1}{} \dots$$

we create a particle with momentum $|p_1\rangle$

we can built up a state

2 quanta in osc. 1

e.g. $|121000\dots\rangle = \left[\frac{1}{\sqrt{2!}} (a_1^\dagger)^2\right]\left[\frac{1}{1!} a_2^\dagger\right]|0\rangle$

1 quantum in oscill. 2

so we can think of this situation as

$a_{p_m}^\dagger$ creates a particle with momentum p_m $|p_m\rangle$

But we need to think of

→ Indistinguishability & SYMMETRY

What I want to do is to repeat the same consideration about sym. and antisym. argument for bosons and fermions.

e.g. we have $|p_1\rangle$ and $|p_2\rangle$ to describe the occupation number $|n_1 n_2\rangle$

$a_{p_1}^\dagger |0\rangle = |10\rangle \qquad a_{p_2}^\dagger |0\rangle = |01\rangle$

Lets add another particle into the vacuum:

$$a_{p_2}^\dagger a_{p_1}^\dagger |0\rangle \propto |11\rangle$$

$$a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle \propto |11\rangle \qquad \Rightarrow$$

$$a_{p_1}^\dagger a_{p_2}^\dagger = \lambda\, a_{p_2}^\dagger a_{p_1}^\dagger \quad \Rightarrow \lambda = \pm 1$$

As before we select:

$\lambda = +1$ = bosons

$$a^+_{p_2} a^+_{p_1} = a^+_{p_1} a^+_{p_2} \rightarrow [\ ] = 0$$

$$[a_i \; a^+_j] = \delta_{ij}$$

those commutation rules are the same as for oscillators.

The many particle state of bosons:

$$|n_1 \; n_2 \; \dots\rangle = \prod_m \frac{1}{(n_{p_m}!)^{1/2}} (a^+_{p_m})^{n_{p_m}} |0\rangle$$

$$a^+_{p_1} a^+_{p_2} |0\rangle = a^+_{p_2} a^+_{p_1} |0\rangle = |1_{p_1} 1_{p_2}\rangle$$

in general
$$\begin{cases} a^+_i |n_1 \dots n_i \dots\rangle = \sqrt{n_i+1}\, |n_1 \dots n_i+1 \dots\rangle \\ a_i | \dots n_i \dots\rangle = \sqrt{n_i}\, | \dots n_i - 1\rangle \end{cases}$$

FERMIONS

Case 2: $\lambda = -1$ $\Rightarrow$

$$\{c_i^+ \; c_j^+\} \equiv c_i^+ c_j^+ + c_j^+ c_i^+ = 0$$

anticommutator

if I set $i \neq j$ $\longrightarrow$ $c_i^+ c_i^+ = 0$ (no way for fermions)

$$c_i^+ |n_1 \dots n_i \dots\rangle = (-1)^{\Sigma_i} \sqrt{1-n_i}\, |n_1 \dots n_i+1 \dots\rangle$$

$$c_i | \dots n_i \dots\rangle = (-1)^{\Sigma_i} \sqrt{n_i}\, | \dots n_i - 1 \dots\rangle$$

$$(-1)^{\Sigma_i} \equiv (-1)^{n_1+n_2+n_3 \dots n_{i-1}}$$

$\longrightarrow$ PAULI exclusion principle.

and $c_i^+ c_j^+ = -c_j^+ c_i^+$ $\Rightarrow$ it matters at what order you place fermions into the states!

Check that $n_i = c_i^+ c_i$ works!

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what if we move from the box to the continuum $\Rightarrow$ or very large box

The continuum limit

1) $\delta_{ij} \longrightarrow \delta^3(p)$

As the size of the system goes up spacing in p goes very much down. $\{\Delta p \cdot \Delta x \sim \hbar\}$

$[a_p\, a_q^\dagger] = \delta^3(p-q)$ and

$$H = \int d^3p\, \varepsilon_p \underbrace{a_p^\dagger a_p}_{\equiv n_p}$$

e.g. For a single-particle state

$$\langle p | p' \rangle = \langle 0 | a_p a_{p'}^\dagger | 0 \rangle$$

$$\langle p|p'\rangle = \langle 0 | [\delta^3(p-p') \pm a_{p'}^\dagger a_p] | 0 \rangle =$$

$$= \langle 0 | \delta^3(p-p') | 0 \rangle = \delta^3(p-p')$$

So it works and we can rewrite both operators and states in terms of the number of particles with momentum p and the very special state $|0\rangle$.

SUMMARY:

- The occupation number representation describes states by listing the number of identical particles in each quantum state.
- We focus on the vacuum state $|0\rangle$ and then construct many-particle states by acting on $|0\rangle$ with creation operators.
- To obey the symmetries of many-particle mechanics, bosons are described by commuting operators and fermions are described by anticommuting operators.