PHYS 502 2018 - Home Work 2 Solutions

Problem 1.

1.1 A harmonic oscillator potential is subjected to the perturbation $\lambda b x^2$ in the time between 0 to T. Obtain the selection rules for the transition from the initial state ϕ_i to ϕ_f in time T and the transition probabilities for the possible transitions.

The selection rule for allowed transitions is $H^{(1)}_{fi} \neq 0$. For $H^{(1)} = bx^2$ we obtain

$$\begin{aligned} H^{(1)}_{fi} &= \left\langle \phi_f \left| H^{(1)} \right| \phi_i \right\rangle \\ &= b\left\langle f \left| x^2 \right| i \right\rangle \\ &= \frac{b\hbar}{2m\omega} \left\langle f \left| (a + a^\dagger)^2 \right| i \right\rangle \\ &= \frac{b\hbar}{2m\omega} \left\langle f \left| a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2} \right| i \right\rangle \\ &= \frac{b\hbar}{2m\omega} \left[\left\langle f \left| \sqrt{i(i-1)} \right| i-2 \right\rangle + \langle f | i+1 | i \rangle + \langle f | i | i \rangle \right. \\ &\qquad \left. + \left\langle f \left| \sqrt{(i+1)(i+2)} \right| i+2 \right\rangle \right] \\ &= \frac{b\hbar}{2m\omega} \left[\sqrt{i(i-1)}\, \delta_{f,i-2} + (2i+1)\delta_{fi} \right. \\ &\qquad \left. + \sqrt{(i+1)(i+2)}\, \delta_{f,i+2} \right] . \end{aligned}$$

For allowed transitions $H^{(1)}_{fi} \neq 0$. Therefore, we have $f = i - 2$ or $i + 2$. The selection rule is $f = i \pm 2$. Then

$$H^{(1)}_{i-2,i} = \frac{b\hbar}{2m\omega} \sqrt{i(i-1)} \,, \quad H^{(1)}_{i+2,i} = \frac{b\hbar}{2m\omega} \sqrt{(i+1)(i+2)} \,.$$

Substituting $f = i - 2$ and $i + 2$ in the expression for the transition probability given by

$$P_{fi} = \frac{4 \left| H^{(1)}_{fi} \right|^2}{\hbar^2 \omega_{fi}^2} \sin^2(\omega_{fi} T/2)$$

we can obtain the transition probabilities for the transitions from i to $i - 2$ and i to $i + 2$ respectively.

1.2

If the perturbation added to a harmonic oscillator potential is $\lambda b x^3$ find the selection rules and the transition probabilities for the allowed transitions.

We calculate $H_{fi}^{(1)}$:

$$
\begin{aligned}
H_{fi}^{(1)} &= b\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left[\langle f\left|(a+a^\dagger)^3\right|i\rangle\right] \\
&= b\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left[\langle f\left|a^3+a^2a^\dagger+aa^\dagger a+aa^{\dagger 2}+a^\dagger a^2\right.\right. \\
&\qquad \left.\left.+a^\dagger a a^\dagger+a^{\dagger 2}a+a^{\dagger 3}\right|i\rangle\right] \\
&= b\left(\frac{\hbar}{2m\omega}\right)^{3/2}\Big[\sqrt{i(i-1)(i-2)}\,\delta_{f,i-3} \\
&\qquad +\sqrt{(i+1)(i+2)(i+3)}\,\delta_{f,i+3} \\
&\qquad +3i^{3/2}\delta_{f,i-1}+3(i+1)^{3/2}\delta_{f,i+1}\Big]\,.
\end{aligned}
$$

$H_{fi}^{(1)}$ is nonzero for $f=i-1$ or $i-3$ or $i+1$ or $i+3$. Therefore, the selection rules are $f=i\pm 1,\ i\pm 3$.

The transition probabilities for the allowed transitions are obtained as

$$
\begin{aligned}
P_{i+1,i} &= \frac{36(i+1)^3 b^2\hbar}{8m^3\omega^5}\sin^2(\omega_{i+1,i}T/2)\,, \\
P_{i-1,i} &= \frac{36 i^3 b^2\hbar}{8m^3\omega^5}\sin^2(\omega_{i-1,i}T/2)\,, \\
P_{i+3,i} &= \frac{(i+1)(i+2)(i+3)b^2\hbar}{18m^3\omega^5}\sin^2(\omega_{i+3,i}T/2)\,, \\
P_{i-3,i} &= \frac{i(i-1)(i-2)b^2\hbar}{18m^3\omega^5}\sin^2(\omega_{i-3,i}T/2)\,.
\end{aligned}
$$

A direct inspection of the solutions in 1.1 and 1.2 shows that of the potential with x^n the selection rules are *f=i+(n-2)* or *i-(n-2)* or *i+n* or *i-n.*

Problem 2.

At time $t = 0$ the infinite height potential $V(x) = 0$ for $0 < x < L$ and ∞ otherwise is perturbed by the additional term of the form $V_{\rm p}(x) = V_0$ for $L/4 < x < 3L/4$ and 0 otherwise. The perturbation is switched-off at $t = T$. The system is initially in the ground state ϕ_1. What is the probability of finding it in the state ϕ_3 after the time $t = T$?

The energy eigenvalues and eigenfunctions of the unperturbed system are

$$E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{2mL^2} , \quad \phi_n^{(0)} = \sqrt{2/L}\,\sin(n\pi x/L) .$$

From the time-dependent perturbation theory the probability of finding the system in ϕ_f at time T if the system is at ϕ_i at $t = 0$ is given by

$$P_{fi} = a_f^* a_f = \frac{4\left|H_{fi}^{(1)}\right|^2}{\hbar^2\omega_{fi}^2}\sin^2(\omega_{fi}T/2) , \quad H_{fi}^{(1)} = \int_0^L \phi_f^{(0)*} H^{(1)} \phi_i^{(0)}\,{\rm d}x ,$$

where $\omega_{fi} = (E_f^{(0)} - E_i^{(0)})/\hbar$. For the given problem

$$P_{31} = \frac{4\left|H_{31}^{(1)}\right|^2}{\hbar^2\omega_{31}^2}\sin^2(\omega_{31}T/2) , \quad \omega_{31} = \frac{E_3^{(0)} - E_1^{(0)}}{\hbar} = \frac{4\pi^2\hbar}{mL^2} .$$

$H_{31}^{(1)}$ is obtained as

$$\begin{aligned} H_{31}^{(1)} &= \frac{2V_0}{L}\int_{L/4}^{3L/4} \sin(3\pi x/L)\,\sin(\pi x/L)\,{\rm d}x \\ &= \frac{V_0}{L}\int_{L/4}^{3L/4} [\cos(2\pi x/L) - \cos(4\pi x/L)]\,{\rm d}x \\ &= -\frac{V_0}{\pi} . \end{aligned}$$

Therefore,

$$P_{31} = \frac{V_0^2 L^4 m^2}{4\pi^6\hbar^4}\sin^2(\omega_{31}T/2) .$$

Problem 3.

Assume that an adiabatic perturbation of the form $H^{(1)} = W(x)\mathrm{e}^{\alpha t}$ is turned on slowly from $t = -\infty$. Obtain the expression for second-order transition amplitude. Also write the time-independent wave function upto second-order correction.

We have the second-order correction term

$$a_f^{(2)} = \frac{1}{(\mathrm{i}\hbar)^2}\sum_n \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \mathrm{e}^{\mathrm{i}\omega_{fn}t''} H_{fn}(t'')\, \mathrm{e}^{\mathrm{i}\omega_{ni}t'} H_{ni}(t') \quad (14.1)$$

and

$$H_{fn}(t'') = \left\langle f \left| W(x)\mathrm{e}^{\alpha t''} \right| n \right\rangle = H_{fn}\mathrm{e}^{\alpha t''}\ , \quad H_{ni}(t') = \quad H_{ni}\mathrm{e}^{\alpha t'} (14.2)$$

Substitution of (14.2) in (14.1) gives

$$\begin{aligned} a_f^{(2)} &= \frac{1}{(\mathrm{i}\hbar)^2}\sum_n \int_{-\infty}^{\infty} \mathrm{d}t' \left[\int_{-\infty}^{t'} \mathrm{d}t'' \mathrm{e}^{\mathrm{i}\omega_{fn}t''} H_{fn}\mathrm{e}^{\alpha t''}\right] \mathrm{e}^{\mathrm{i}\omega_{ni}t'} H_{ni}\mathrm{e}^{\alpha t'} \\ &= \frac{1}{(\mathrm{i}\hbar)^2}\sum_n \frac{H_{fn}}{\mathrm{i}(\omega_{fn}-\mathrm{i}\alpha)} H_{ni} \int_{-\infty}^{t} \mathrm{e}^{\mathrm{i}(\omega_{ni}+\omega_{fn}-2\mathrm{i}\alpha)t'}\, \mathrm{d}t'\ . \quad (14.3) \end{aligned}$$

Substituting $\omega_{ni} + \omega_{fn} = \omega_{fi}$ and integrating the above equation we get

$$a_f^{(2)} = \frac{1}{\hbar^2}\sum_n \frac{H_{fn}H_{ni}}{(\omega_{fn}-\mathrm{i}\alpha)(\omega_{fi}-2\mathrm{i}\alpha)}\, \mathrm{e}^{\mathrm{i}(\omega_{fi}-2\mathrm{i}\alpha)t}\ . \quad (14.4)$$

Then

$$\psi = \sum_f a_f(t)\mathrm{e}^{-\mathrm{i}E_f t/\hbar}\phi_f^{(0)}\ , \quad (14.5)$$

where $a_f(t) = a_f^{(0)} + a_f^{(1)} + a_f^{(2)}$. Next, we obtain

$$\begin{aligned} a_f^{(1)} &= \frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{e}^{\mathrm{i}\omega_{fi}t'} H_{fi}(t')\, \mathrm{d}t' \quad = \quad \frac{H_{fi}}{\mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{e}^{\mathrm{i}\omega_{fi}t'}\mathrm{e}^{\alpha t'}\, \mathrm{d}t' \\ &= -\frac{H_{fi}}{\hbar(\omega_{fi}-\mathrm{i}\alpha)}\, \mathrm{e}^{\mathrm{i}(\omega_{fi}-\mathrm{i}\alpha)t}\ . \quad (14.6) \end{aligned}$$

Further, $a_f^{(0)}(t) = \delta_{fi}$. The wave function is given by Eq. (14.5) with a_f's given by (14.6) and (14.4) with $a_f^{(0)}(t) = \delta_{fi}$.

Problem 4.

A one-dimensional harmonic oscillator has its spring constant k suddenly reduced by a factor of $1/2$. The oscillator is initially in its ground state. Find the probability for the oscillator to remain in the ground state after the perturbation.

The transition coefficient $a_f^>$ is given by $\langle \phi_f^> | \phi_i^< \rangle$. We have

$$\begin{aligned}\phi_i^< &= \left(\frac{\alpha_i^2}{\pi}\right)^{1/4} \mathrm{e}^{-\alpha_i^2 x^2/2}, \quad \alpha_i^2 = \frac{\sqrt{km}}{\hbar} \\ \phi_f^> &= \left(\frac{\alpha_f^2}{\pi}\right)^{1/4} \mathrm{e}^{-\alpha_f^2 x^2/2}, \quad \alpha_f^2 = \frac{\sqrt{km/2}}{\hbar} \; .\end{aligned}$$

Now

$$\begin{aligned}a_f^> &= \int_{-\infty}^{\infty} \phi_f^{>*} \phi_i^< \, \mathrm{d}x \\ &= \sqrt{\frac{\alpha_i \alpha_f}{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(\alpha_i^2 + \alpha_f^2) x^2/2} \, \mathrm{d}x \\ &= \left[\frac{2\alpha_i \alpha_f}{\alpha_i^2 + \alpha_f^2}\right]^{1/2} \; .\end{aligned}$$

The probability for the system to remain in the ground state after changing the spring constant by the factor of half is

$$|a_f|^2 = \frac{2\alpha_i \alpha_f}{\alpha_i^2 + \alpha_f^2} = \frac{2\left(\sqrt{k}\sqrt{k/2}\right)^{1/2}}{\sqrt{k} + \sqrt{k/2}} = \frac{2(2)^{1/4}}{1 + \sqrt{2}} = 0.985 \; .$$