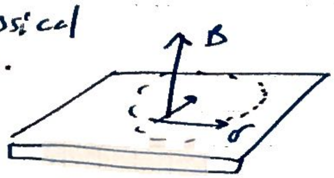
# Fermi Surface and Transport.

Classical Mech.




Consider an electron moving in 2D

$\bar{F} = \frac{q}{c}\bar{v} \times \bar{B}$ ⇒ cause of the circular motion

$$a = \frac{v^2}{r} = \frac{F}{m} = \frac{q}{mc} v \times B$$ and since $v \perp B$

$$v = \frac{q}{mc} B r$$ and the period of the oscillations

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\frac{q}{mc} B r} = \frac{2\pi mc}{qB}$$

and $$\omega = \frac{2\pi}{T} = \frac{qB}{mc}$$

Semiclass. Mechanics: $$F = \frac{dp}{dt} \Rightarrow$$

$$\frac{d\bar{p}}{dt} = \frac{q}{c}\frac{d\bar{r}}{dt} \times \bar{B}$$

$$\frac{dr}{dt} = \nabla \epsilon(\bar{p}) \Leftarrow \frac{dr}{dt} = v = \frac{d\omega}{dk} = \frac{\hbar d\omega}{\hbar dk} = \frac{d\epsilon}{dk}$$

So $\frac{dp}{dt}$ is always perpendicular to $\nabla\epsilon(p)$

- where $\nabla\epsilon(p)$ is the gradient of energy.

By applying this ideas to the 2DEG with magnetic field. The electron will move on the equal energy curve. for $\epsilon > \mu$ (no electrons), for $\epsilon < \mu$ all states are taken. So what matters is only $\epsilon = \mu$.

Quantization:

$$\oint \bar{p} \cdot d\bar{r} = 2\pi\hbar(n+\gamma)$$ the Bohr-Somerfeld condition

$n = 1, 2, 3, \ldots \quad \gamma = 1/2$

$$\Rightarrow \oint p\, dr = \frac{2\pi\hbar}{\lambda}\oint dr = \frac{2\pi\hbar}{\lambda} \cdot 2\pi r = 2\pi\hbar(n+\lambda)$$

ignore $\lambda$ $$\Rightarrow \frac{2\pi r}{\lambda} = n$$

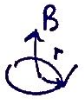
When we move around the circle
we want the circumference $= n \cdot \lambda$

$$\frac{dp}{dt} = \frac{q}{c}\frac{dr}{dt} \times B \Rightarrow$$

$$p = \frac{q}{c} r \times B + \text{const}$$

ignore constant by redefining $r$:

$$p = \frac{q}{c} \bar{r} \times \bar{B}$$



$$r \times B = \frac{q}{c}(r \times B) \times B = \frac{q}{c}\left[B(r \cdot B) - r(B \cdot B)\right]$$

$$= -\frac{q}{c} r \cdot B^2 \Rightarrow r = -\frac{c}{q}\frac{p \times B}{B^2}$$

$$dr = -\frac{c}{q}\frac{dp \times B}{B^2} \Rightarrow \oint p\,dr = -\frac{c}{q}\oint \bar{p}\,\frac{d\bar{p} \times \bar{B}}{B^2} =$$

$$= -\frac{c}{qB^2}\oint \bar{p} \cdot (d\bar{p} \times \bar{B}) = -\frac{c}{qB}\oint p \times dp =$$

$$= -\frac{c}{qB}\hbar^2 \int (k \times dk) = -\frac{c}{qB}\hbar^2 S = 2\pi\hbar(n+\gamma)$$

where $S$ is the area enclosed by the Fermi surface in the $k$-space

$$\frac{1}{B} = -2\pi(n+\gamma)\frac{q}{c\hbar S}$$

So by tuning the magnetic field $B$ and measuring say magnetization as a function of $B$ we should observe oscillations with the periodicity

$$\boxed{\Delta\left(\frac{1}{B}\right) = \frac{2\pi e}{\hbar c S}}$$

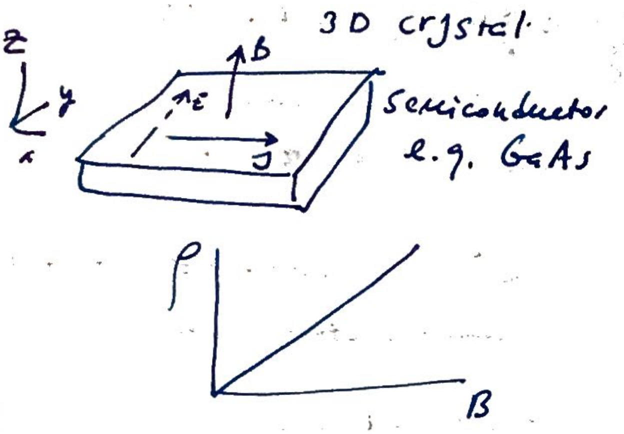
This periodicity depends on the shape of the Fermi surface.

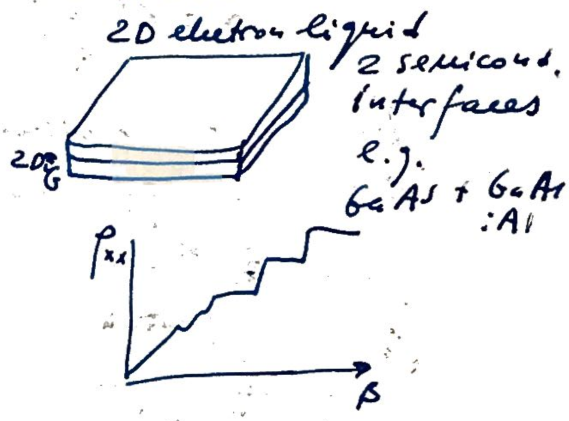
This way you can determine the size and the shape of the Fermi surface pockets of electrons and holes.

See exp. figure

# Quantum Hall effect and Topological insulators

Here is a very interesting observation.




$\rho_{xx} = E_x / j_x$

$\rho_{xy} = E_y / j_x$

Based on the cylindrical symmetry

$$\rho_{xx} = \rho_{yy} \qquad \rho_{xy} = \rho_{yx}$$

From classical mechanics (Debye theory) $\rho_{xy} \sim B$

$$eE_y = evB \qquad j = evn \Rightarrow \rho_{xy} = \frac{E_y}{J_x} = \frac{B}{en}$$

Very useful but nothing too spectacular:

## Quantum Hall effect and its connection to quantum electrodynamics

At low T and very very clean samples the 2DEG dosn't follow $\rho_{xy} \sim B$!

Instead is shows a series of very strange platoes. On the platoe $\sigma_{xy} = \frac{1}{\rho_{xy}}$ is quanties $= \frac{e^2}{h} n$ $\quad n = 1, 2, 3, \ldots$ and $\rho_{xx} = 0$!

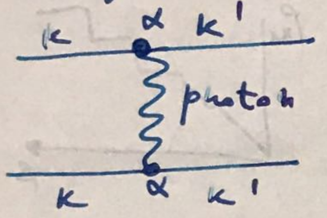
the filling factor.

Those platos are known as the Q. H. E.

Interestingly the value of conductivity can be expressed in terms of the fine structure constant:

$$\alpha = \frac{e^2}{\hbar c} \qquad \sigma_{xy} = n\frac{e^2}{\hbar} \Rightarrow = n \cdot \alpha \cdot c$$

The f.s. constant $\alpha$ measures the strength of quantum electrodynamics:




now $e^2 \sim$ coulomb potential
denominator $\sim c$

$$\alpha \sim \frac{\text{potential energy}}{\text{kinetic energy}}$$

If in our universe $\alpha = 0$ then electrons will not interact and in fact there will be no photons (no light - just imagine this).

If $\alpha$ is large the universe will be made of very strongly entangled matter which will make the presence of life impossible as we know it.

However $\alpha \sim \frac{1}{137}$ which is just 1% of the kinetic energy and we leave in the world with interactions ≡ photons ≡ light and yet we can do the perturbation theory.

Consiter just kinetice energy as the unperturbed term, then in the 1st approx

$O(\alpha) \sim 1\%$, $O^2(\alpha^4) \sim 10^{-8}$ !

Amazing but a condensed matter experiment can be as accurate as high energy physics in defining $\alpha$!

QED:

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

CMP: $\alpha_{CMP} = \frac{e^2}{\hbar v_{Fermi}}$ for a typical solid $v_F \sim \frac{1}{100} - \frac{1}{1000} c$

So $\alpha_{CMP} = \frac{e^2}{\hbar v_F} \sim 1 \div 10$

so the perturbation theory don't work.

## QHE and Topology.

Before we answer why $\sigma_{xy}$ is quantized
lets try to think why $\sigma_{xx} = 0$.

Q: if $\rho_{xx} = 0$ is it a superconductor / or "perfect metal

A: NOooooo.. The material is
~~SC~~ - ~~Conductor~~ - Insulator!
wow....

The material has 0 conductivity.

$j = \sigma E$ , $E = \rho j$ $\rho = \frac{1}{\sigma}$

This is only true if $j$ and $E$ are in the same direction. However more generally:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \cdots & \sigma_{xz} \\ \vdots & \ddots & \vdots \\ \sigma_{zx} & \cdots & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

if all but $\sigma_{xx}$ $\sigma_{yy}$ $\sigma_{zz}$ are not zero
$j = \sigma E$ and $\rho = \frac{1}{\sigma}$ but for 2D

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$ and the resistivity

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}^{-1} =$$

$$= \frac{1}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \begin{pmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{pmatrix}$$

Now lets go to to the platoe where $\rho_{xx} = \rho_{yy} = 0$

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \rho_{xy} \\ \rho_{yx} & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{1}{-\rho_{xy}\rho_{yx}} \begin{pmatrix} 0 & -\rho_{xy} \\ -\rho_{yx} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/\rho_{yx} \\ 1/\rho_{xy} & 0 \end{pmatrix} \Rightarrow \sigma_{xx} = \sigma_{yy} = 0$$ NO CONDUCTIVITY

so the system is INSULATOR.

But between the platoes the conductivity is non zero and as such it's METAL.

$\rho_{xx} \neq 0$ and $\sigma_{xx}$ and $\sigma_{yy} \neq 0$.

## SUMMARY OF EXP. FACTS ABOUT QHE.

1. When we vary the external field
- we can turn system ... ins ⇔ metal ⇔ ins. ⇔ metal ...
- each insulating state corresponds to a plateau of $\rho_{xy}$ and the step between 2 neighboring plateaus is metallic.
- transport for the metalic states is NOT universal, and changes from sample to sample.
- The insulating state is UNIVERSAL. $\rho_{xx} = 0$ and $\sigma_{xx} = 0$ and $\sigma_{xy}$ is quantized.

• WHY IQH state is insulator?

LANDAU LEVELS.

Lets solve this problem quantum mechanically. In 2D for a charge neutral particle $q=0$. the Schrödinger eqn:

$$i \frac{\partial \psi(x,y)}{\partial t} = \left[ \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial y} \right)^2 \right] \psi(x,y)$$

with $H = \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial y} \right)^2$

Now lets include charge and its connection to E and B. He we will use a minimal coupling which tells that we change momentum $\vec{p} \rightarrow \vec{p} + \frac{e\vec{A}}{c}$ $A$ = vector potential

and $i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} - \frac{e}{c}\phi$ $\phi$ = electric potential

$$\left(i\frac{\partial}{\partial t} - \frac{e}{c}\Phi\right)\Psi(x,y) = \left[\frac{1}{2m}\left(-i\hbar - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2\right]\Psi(x,y)$$

$$i\frac{\partial}{\partial t}\Psi(x,y) = \left\{\left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2\right] + \frac{e}{c}\Phi\right\}\Psi(x,y)$$

$$H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2 + \frac{e}{c}\Phi$$

Since $E=0$ we can set $\Phi = 0 \Rightarrow$

$$\nabla\times A = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = B$$

from E & M we know that A is not observable i.e. for the fixed B field, A is not uniquely defined. $\nabla\times A = B$ $A' = A + \nabla\chi$ is also the vector potential for the same field $\nabla\times A' = B$. We can choos any A as long as as $\nabla\times A = \nabla\times A'$. Now lets apply the magnetic field along z: $B_z$ Then we can use 2 options: $A_x = 0$ $A_y = Bx$ (the Landau gage)

$A_x = -\frac{By}{2}$ and $A_y = \frac{Bx}{2}$ (the symmetric gage)

$$\left.\begin{aligned}\frac{\partial}{\partial x}A_y &= B\\ \frac{\partial}{\partial x}A_x &= 0\end{aligned}\right\}\ \nabla\times A = B.$$

## Energy spectrum

$$H = \left[\frac{\hbar^2}{2m}\left(-i\frac{\partial}{\partial x}\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}Bx\right)^2\right].$$

Static Sch. equation: $H\Psi = E\Psi$

in the Landau gage $[p_y, H] = 0$ therefore we can find common eigen states for $p_y$ and $H$.

$$\Psi(x,y) = f(x)\, e^{-ik_y y}$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{1}{2m}\left(\hbar k_y - \frac{e}{c}Bx\right)^2 f(x) = \epsilon f(x)$$

(for $p_y$ the eigenvalue $\hbar k_y$), lets rewrite it:

$$-\frac{\hbar^2}{2m} f''(x) + \frac{e^2}{2mc^2}B^2\left(x - \frac{c\hbar}{eB}k_y\right)^2 f(x) = \epsilon f(x)$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{K}{2}(x - x_0)^2 f(x) = \epsilon f(x)$$

where $x_0 = \frac{c\hbar}{eB}k_y$ $\quad K = \frac{e^2B^2}{mc^2}$; $x_0 = \ell_c^2 k_y$ where $\ell_c^2 = \frac{c\hbar}{eB}$

now this equation looks like the harmonic oscillator:

Recall from your QM class, the solution is

$$\begin{cases} \Psi_{n,k_y}(x,y) = \phi_n(x - x_0)\, e^{-ik_y y} \quad \text{with} \\ \epsilon_{n,k_y} = \left(n+\frac{1}{2}\right)\hbar\omega = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{K}{m}} = \left(n+\frac{1}{2}\right)\underbrace{\frac{eB\hbar}{cm}}_{\omega_c} \end{cases}$$

$n = 0, 1, 2, 3 \ldots$

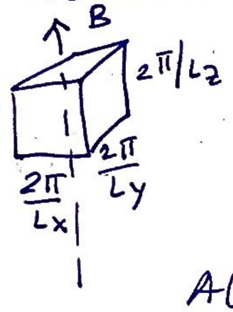
So the electrons are quantized in $x$-$y$ and have continious translation along $z$; so the total $E$

$$E = E(n, k_y) + \frac{\hbar^2}{2m}k_z^2 = \left(n+\frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2}{2m}k_z^2$$

If we include the spin of the electron then each level will split into 2 sublevels $\pm g_e \mu_B \sigma H$

Since the energy spectrum is dramatically affected we need to see what happens to the electronic density of states.

We start with




$\Rightarrow$ ? $\Rightarrow$ $k_x$ and $k_y$ are quantized in units $\frac{2\pi}{L_x}$ and $\frac{2\pi}{L_y}$

Also recall $x_0 = \frac{c\hbar}{eB} k_y = \frac{2\pi \ell_c^2}{L_y}$

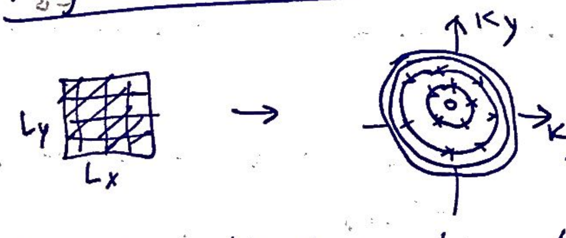
And the degeneracy of the level in 2D is:

$D = \frac{L_x}{\Delta x_0} = \frac{L_x L_y}{2\pi \ell_c^2}$ ; The total magnetic flux through the x-y plane is

$\Phi = H L_x L_y$ and the flux quantum: $\phi_0 = \frac{\hbar c}{e} \Rightarrow$

$D = \frac{\Phi}{\phi_0} \overset{or}{\Rightarrow}$ the number of states = number of the flux quanta in units of $\frac{\hbar c}{e}$!

The Physical Meaning:



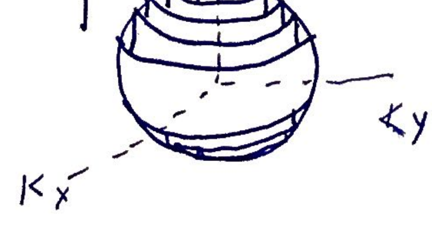

After applying the magnetic field points distributed in the $k_x - k_y$ area get spread out onto concentric circles with energies $\frac{\hbar\omega_c}{2}, \frac{3}{2}\hbar\omega_c, \frac{5}{2}\hbar\omega_c \ldots$

But the total number of states remain the same. To show this lets calculate the number of states per unit area per unit energy and no spin

$$g(E) = \frac{1}{L_x L_y}\frac{D}{\hbar\omega_c} = \frac{m}{2\pi\hbar^2}$$

= which is the same as = $g^{2D}(E)$ without magnetic field attached.

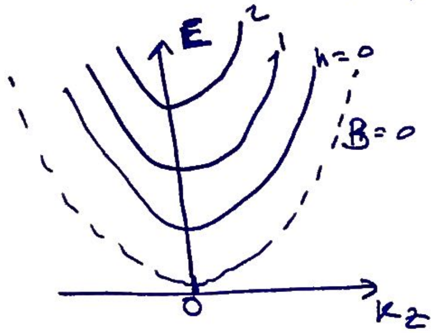
Now lets extend this to 3D:

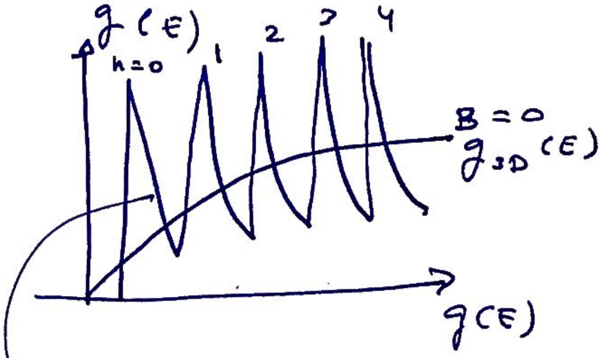



Note $k_z$ is still a good quantum number so we can plot $E(k_z)$ vs. $k_z$ as bands also known as the Landau subbands

Overall for 3D we have:

$$g_{3D}(E) = \frac{1}{(4\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_n \left[E - (n+\tfrac{1}{2})\hbar\omega_c\right]^{-1/2}$$

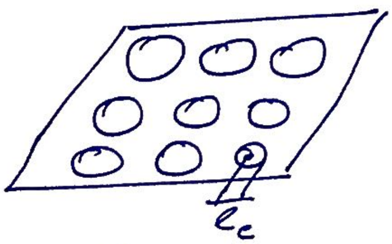


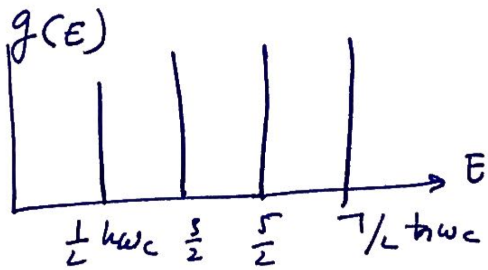




1D spikes from the states on the circumference of the circles.

$$g_{1D}(E) = \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} (E-E_n)^{-1/2}$$

For 2D this is very interesting



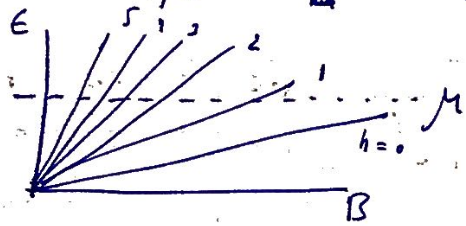




If we apply E along x-direction we going to have the famous QHE.

~~[illegible]~~

Landau levels as energy bands

$\epsilon_{n,k} \sim B(n+\frac{1}{2})$




– The lines = Landau levels, where $n$ is the band index & $k_y$ is the momentum.

– All states with the same $n$ but different $k$ have the same energy. The bands are FLAT.

– if $\mu$ is in the gap the material is insulator. So we are on the QHE plateaus; or if $\mu$ crosses the band it's a metal.

If in the experiment we fix $\mu$ (the number of electrons) in the system and vary $B$. We change the energy of all levels.

EDGE STATES and quantization OF HALL CONDUCTIVITY.

I) Lets start from the semiclassical picture.

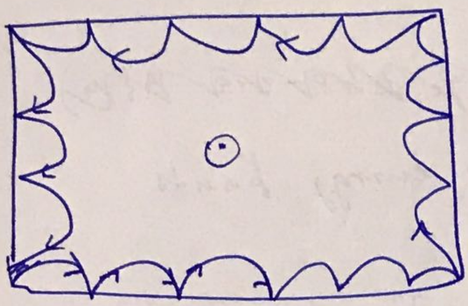
$F = \frac{q}{c} v \times B$ and going through the same arguments as before we get:

$$\omega_c = \frac{2\pi}{T} = \frac{qB}{mc} \Rightarrow E = \hbar\omega = \frac{\hbar q B}{mc}$$

which is the energy separation between 2 ~~[illegible]~~ consequitive Landau Levels.

CHIRAL EDGE STATES

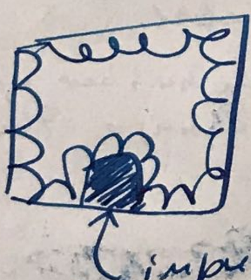
Now we are close to the edge



The electrons are moving only in 1 direction ~~and~~ which is defined by the direction of the applied field.

We can consider the edge ~~to~~ a 1D wire.
- in 1D wire electrons can move both ways
- in the QHE we can only move one way

This "one-way" or handidness plays an important role if you consider the impurity, i.e


impurity potential

DISPERSION RELATIONSHIP OF THE CHIRAL STATES

$v = \frac{dw}{dk} = \frac{\hbar\, dw}{\hbar\, dk} = \frac{dE}{\hbar\, dk}$ For edge state electron moves in 1D.

Since electron moves to the right $v > 0 \rightarrow \frac{dE}{dk} > 0$

Since $\frac{dE}{dk} > 0$ $E$ increases monotonically with $k$.

We only focus on $E \sim E_{Fermi}$. Lets expand it around the fermi edge:

$$E(k) = E_F + v_F \hbar (k - k_F) + \ldots$$

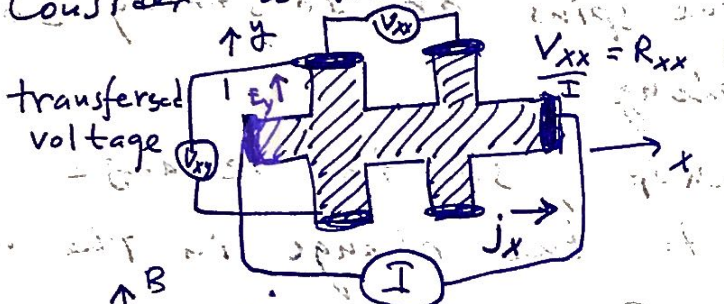
$v_F = \frac{dE}{dk} \cdot \frac{1}{\hbar} \Big|_{k=k_F}$ ; for righ moving particles $v_F > 0$

left $v_F < 0$

and $E(k)$ decreases with $k$

# Hall CONDUCTIVITY

Consider a Hall bar here.





$v_F < 0$ Left moving on the top surface

$v_F > 0$ on the bottom surface

Case 1: $E_y = 0$ Top and bottom edges have the same electrical potential. So the number of electrons moving to the left = to the right. The total current = 0. in x-direction.

Case 2: $E_y \neq 0$. Top and the bottom have different potentials. Top $+V/2$ and the bottom $-V/2$ since top is positive it attracts move electrons($e^-$) so the # of electrons moving left $\neq$ right to the left. Which means So we have net flow of $e^-$ that current goes to the right.

To compute the current: Let's compute the number of electrons attracted to the top:

The energy of electrons on the bottom:

$$\epsilon = E_F + v_F \hbar (k - k_F) + eV/2$$

since the electron carries charge $-e$ the potential energy $(-e) \times (-V/2) = eV/2$. Since the potential term the $k_F$ changes but $\epsilon = E_F$ remains.

$$\epsilon = E_F + v_F \hbar (k - k_F) + eV/2 = E_F \quad \hbar v_F (k - k_F) = -eV/2$$

$k = k_F - eV/(2\hbar v_F)$, so after the potential

$k < k_F - eV/(2\hbar v_F)$ as opposed to the case with $E_y = 0$ $k < k_F$

The number of removed electrons:

$$N_b = \int_{k_F - eV/2}^{k_F} \frac{dk}{2\pi/L_x} = \frac{L_x}{2\pi}\left[k_F - \left(k_F - \frac{eV}{2v_F\hbar}\right)\right] =$$

$$= L_x/2\pi \, eV/2 = \frac{L_x}{2\pi}\frac{eV}{2v_F\hbar}$$

Notice we don't have the factor 2 refering to spin. I assume the spins are so strong so they all polarized along B.

The current $j = -en v_F$, so the change of current $\delta j$ due to the change in the number of electrons $\delta j = -e \underbrace{\delta n} v_F$.

$\delta n = -\frac{N_b}{L_x L_y}$ so we have

$$\delta j_x^{(bottom)} = (-e)\left(\frac{N_b}{L_x L_y}\right) v_F = e \frac{1}{2\pi} \frac{eV}{2 v_F L_y \hbar} v_F =$$

$$= \frac{e^2}{\hbar} \frac{V}{2L_y} = \frac{e^2}{\hbar} E_y/2 \qquad E_y = \frac{V}{L_y}$$

Similar to the top:

$$\delta j_x^{(t)} = \frac{e^2}{\hbar} \frac{E_y}{2}$$

And the total contribution:

$$\delta j = j_x^{(t)} + j_x^{(b)} = \frac{e^2}{\hbar} E_y$$

Thus $\sigma_{xy} = \frac{j_x}{E_y} = \frac{e^2}{\hbar} \approx 25 k\Omega$

For one chiral edge state we have $\sigma_{xy} = \frac{e^2}{\hbar}$ and for $n = n e^2/\hbar$, $n$ must be integer. It can be shown that the # of chiral states = number of filled Landau Levels.

Q. Great, but what it has to do with topology?