Quantum Hall effect

Bachelor of Science Degree in Physics

Simon Taylor

Supervisor Anton Nedelin

Subject reader Joseph Minahan

Spring 2015

Theoretical Physics
Department of Physics and Astronomy
Uppsala University

Abstract

The main goal of this project was to write a review about different quantum Hall effects. This review focuses on the integer and relativistic quantum Hall effect in graphene. The quantum Hall effect is a newly discovered phenomena that was experimentally observed in 1980 and relativistic quantum Hall effect in graphene was observed in 2005. This project takes a theoretical approach to describe the quantum Hall effects and graphene itself. Experiments has shown that for very strong magnetic fields applied to 2D systems, the Hall resistance becomes quantized, $R_H = h/ne^2$ and only depends on the charge of the electron and Planck's constant, two fundamental constants of nature. This sets a new standard on how to define resistance, and gives a good tool for precise measurements of fine structure constant.

Sammanfattning

Målet med det här projektet är att göra en litteraturstudie om olika kvant-Halleffekter. Den här litteraturstudien fokuserar på heltals och relativistiska Halleffekten i grafen. Kvant-Halleffekten är ett nyupptäckt fenomen som observerades experimentellt 1980 och den relativistiska Halleffekten observerades 2005. Den här litteraturstudien tar en teoretisk inriktning på att beskriva kvant-Halleffekten och grafen. Experiment har visat att vid väldigt starka magnetfält i tvådimensionella system, så blir det elektriska motståndet kvantiserat, $R_H = h/ne^2$ och beror endast på elektronens laddning och Plancks konstant, två fundamentala naturkonstanter. Detta sätter en helt ny standard som definierar det elektriska motståndet.

För att beskriva kvant-Halleffekten så måste man först gå igenom Landaukvantisering, det vill säga att när man applicerar ett starkt magnetfält vinkelrätt mot det tvådimensionella planet åker elektronerna runt i slutna banor, på grund av Lorentzkraften. Dessa banor blir då kvantiserade vilket kallas Landaunivåer.

Contents

1 Introduction **3**

2 Background **3**
2.1 Classical Hall effect 4
2.2 Shubnikov-de Haas effect 8

3 Landau Quantization **9**
3.1 Non relativistic Hamiltonian for a 2D free particle in a zero B-field . . 9
3.2 Hamiltonian for a non zero B-field 10
3.3 Quantum mechanical interpretation 11
3.4 Non relativistic Landau Levels 12
3.5 Relativistic Landau levels 15
3.6 Level degeneracy 16
3.7 Semi-classical interpretation of the level degeneracy 18
3.8 Eigenstates and the wave function in the symmetric and Landau gauge 20

4 Integer Quantum Hall effect **23**
4.1 Electronic motion in an external electrostatic potential 23
4.2 Semi-classical approach 23
4.3 Electrostatic potential with translation invariance in the x-direction . 25
4.4 Landau levels and conductance 25
4.5 Edge states 27
4.6 Integer quantum Hall effect and percolation 27

5 Relativistic Quantum Hall Effect in Graphene **28**
5.1 Graphene 29
5.2 Relativistic Quantum Hall effect 32

6 Conclusions **33**

7 Acknowledgments **34**

1 Introduction

This project is a review about different descriptions of the integer and relativistic quantum Hall effect in graphene. First some introduction to the classical Hall effect will be presented. Then some aspects about Landau quantization will be covered before discussing quantum Hall effects. There are many different quantum Hall effects that has been discovered during the years. For example, quantum integer or fractional, charge or spin or relativistic and non relativistic Hall effects. This project mainly focuses on integer and relativistic Hall effects in graphene.

Edwin Hall discovered an interesting phenomena when one applies an electric field to a metal and at the same time apply a magnetic field in the perpendicular direction. Then Hall showed that the transverse resistance R_H is equal to $R_H = B/qn_{el}$, which is a very interesting result. A result that has some very useful applications in many fields. Quantum Hall effect is a quantum mechanical generalization of the classical Hall effect, which is observed for two-dimensional systems at very low temperatures. Quantum Hall effect is a relativity new discovered phenomena and one of the most interesting phenomena within condensed matter physics.

In 1980 a quantized version of the Hall effect was discovered by Klitzing, Dorda and Pepper, which later gave them the Nobel prize [1, p. 13]. When one create low temperatures and a strong magnetic field in a 2D system (see figure 2), the Hall resistance becomes quantized,

$$R_H = \frac{h}{ne^2}$$

where n is an integer.

Instead for being linearly proportional to the magnetic field, the Hall resistance shows some plateaus (see figure 1). One also notice that the Hall resistance is independent of the properties of the material and is only depended on the Planck's constant, h and the charge of the electron, e, two fundamental constants of nature. This has resulted in a new standardization of resistance which has been used since 1990, $R_{K-90} = h/e^2 = 25812.807\Omega$ [1, p. 14]. It also provides a good tool for measuring the fine structure constant [8].

In graphene, electrons start to behave as massless relativistic particles and obey Dirac equation of motion in two-dimensions. Landau quantization of electron orbits then results in the resistance taking values of $n = \pm 2(2m+1)$, where m is an integer. The different signs corresponds to electron and hole conductivity. All of this will be discussed in greater detail in this review.

2 Background

Before introducing Quantum Hall effect, one may take a look at the classical Hall effect.

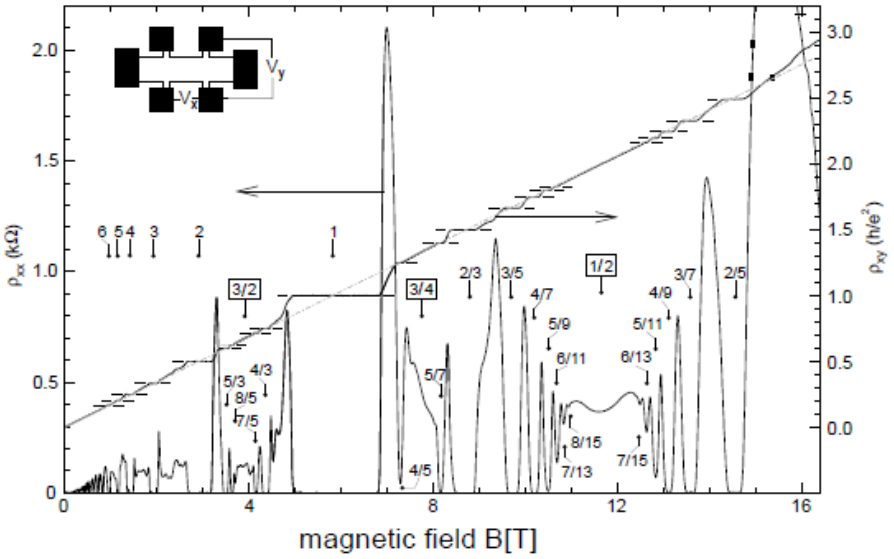

Figure 1: Here is a measurement made by J. Smet, MPI-Stuttgart of quantum Hall effects, both Integer and fractional Hall effect. Here one can see the plateau pattern and at these plateaus the longitudinal resistance goes to zero [1].

2.1 Classical Hall effect

The classical Hall effect was discovered back in 1879 by Edwin Hall. What he found was that the transverse resistance R_H is linearly proportional to a perpendicular magnetic field applied [1, p. 8],

$$R_H = \frac{B}{qn_{el}}, \tag{1}$$

where q is the charge carrier and n_{el} is the 2D carrier density.
Classical Hall effect is basically a phenomena that occurs when one have a metal and apply an electric field such that a current flows along the sample. If one then applies a magnetic field perpendicular to the electric field, the electrons will be effected in such a way that they accumulate at one side of the sample, while opposite charges accumulate on the other side. Then they will create a potential across the sample, called Hall voltage.
To understand this in more detail and come up with equation (1), one has to make some certain assumptions and look at the Hall effect in the framework of the Drude model. The Drude model assumes the electrons to behave as a classical gas within a solid [3, p. 4] and it also assumes that electrons do not interact with each other. Electrons can collide with an ion, which then will result in velocity change of the electron. But meanwhile the electron is traveling, there are no other interactions taking place, due to the free electron approximation [3, p. 5]. Through the collisions, the electrons reach thermal equilibrium with the lattice. Final important approximation is the relaxation time τ. With τ, one can for example define the mean free path of an electron: $\lambda = \tau v_D$ [3, p.6].

If one now go to the case where one has a metal with an electric field applied

to it, one will then see a current going through the sample. This motion can be described by the equation of motion for an electron [2, p. 73]

$$\frac{d\vec{v}}{dt}m_e = -e\vec{E}, \tag{2}$$

which has a solution

$$\vec{v}(t) = \frac{-e\vec{E}t}{m_e}. \tag{3}$$

But due to the collisions with ions that happens after a time τ, the average drift velocity becomes

$$\vec{v}(t) = \frac{-e\vec{E}\tau}{m_e}. \tag{4}$$

To find the charge density, one can imagine a cross sectional area in which the current goes through, then the number of charges going through that area is

$$I = -en|\vec{v}|A \tag{5}$$

then the current density is

$$\vec{j} = -en\vec{v}. \tag{6}$$

Inserting equation (3) to equation (6) gives,

$$\vec{j} = \frac{ne^2\tau}{m_e}\vec{E} \tag{7}$$

and as a result the conductivity is [2, p. 74]

$$\sigma_0 = \frac{ne^2\tau}{m_e} \tag{8}$$

from Ohm's law. This will later be used when deriving equation (1).
Let's now derive the Hall resistance using the Drude model. The momentum of a particle at a given time is $\vec{p}(t)$ and the momentum at an infinitesimal time later is $\vec{p}(t+dt)$ [3, p. 10]. The probability for an electron not to collide during this time is $1 - dt/\tau$. Assuming that the particle is influenced by a force which also change in time $\vec{f}(t)$ and as an infinitesimal time later the particle has increased its momentum and will therefore gain an extra term $\vec{f}(t)dt + \mathcal{O}(dt^2)$. Now considering only the fractions of electrons that do not collide during time τ, the momentum can then be expressed as [3, p. 11]

$$\vec{p}(t+dt) = \left(1 - \frac{dt}{\tau}\right)\left[\vec{p}(t) + \frac{dt}{\tau}\vec{f}(t)dt + \mathcal{O}(dt^2)\right]. \tag{9}$$

Since the electrons move randomly and only acquire additional momentum from the last collision in a time dt, higher order terms than $\vec{f}(t)dt$ will not contribute and hence the momentum change can be written as

$$\vec{p}(t+dt) = \vec{p}(t) - \frac{dt}{\tau}\vec{p}(t) + \vec{f}(t)dt + \mathcal{O}(dt^2). \tag{10}$$

Then dividing by dt and taking the limit as dt goes to zero one will end up with

$$\lim_{dt\to 0}\frac{\vec{p}(t+dt)-\vec{p}(t)}{dt}=\frac{d\vec{p}(t)}{dt}=\lim_{dt\to 0}-\frac{\vec{p}(t)}{\tau}+\vec{f}(t). \tag{11}$$

Finally the equation of motion is [3, p. 11]

$$\frac{d\vec{p}(t)}{dt}=-\frac{\vec{p}(t)}{\tau}+\vec{f}(t). \tag{12}$$

In this case $\vec{f}(t)$ is already known, it is the Lorentz force, since the electrons are influenced by a magnetic and electric field. For the Hall effect, the equation of motion is [1, p. 9]

$$\frac{d\vec{p}(t)}{dt}=-\frac{\vec{p}(t)}{\tau}-e\Big(\vec{E}+\frac{\vec{p}}{m_b}\times\vec{B}\Big). \tag{13}$$

To find the resistivity one has to find the steady state solution where the system is in equilibrium, which is done by setting $d\vec{p}/dt=0$. One should also keep in mind that the system is only in two dimensions, therefore the momentum has components $\vec{p}=(p_x,p_y)$. Setting up the equation of motion in each direction one get [1, p. 9]

$$eE_x=-\frac{eB}{m_b}p_y-\frac{p_x}{\tau} \tag{14}$$

$$eE_y=\frac{eB}{m_b}p_x-\frac{p_y}{\tau} \tag{15}$$

and the first terms of the equations are defined as the cyclotron frequency,

$$\omega_c=\frac{eB}{m_b}. \tag{16}$$

If one then multiplies equation (14) and (15) with the conductivity derived previously (equation (8)) one get

$$\sigma_0E_x=-\frac{\omega_c\sigma_0}{e}p_y-\frac{\sigma_0}{\tau e}p_x \tag{17}$$

$$\sigma_0E_y=\frac{\omega_c\sigma_0}{e}p_x-\frac{\sigma_0}{\tau e}p_y \tag{18}$$

and plugging in equation (8) on the right hand side of equation (17) and (18)

$$\sigma_0E_x=-\frac{en_{el}}{\tau e}p_x-\frac{\omega_cn_{el}e\tau}{m_b}p_y \tag{19}$$

$$\sigma_0E_y=\frac{\omega_cn_{el}e\tau}{m_b}p_x-\frac{en_{el}}{\tau e}p_y. \tag{20}$$

Then one can rewrite the system of equations in terms of current density, which is defined as [1, p. 9]

$$\vec{j}=en_{el}\vec{v}=-\frac{en_{el}\vec{p}}{m_b}, \tag{21}$$

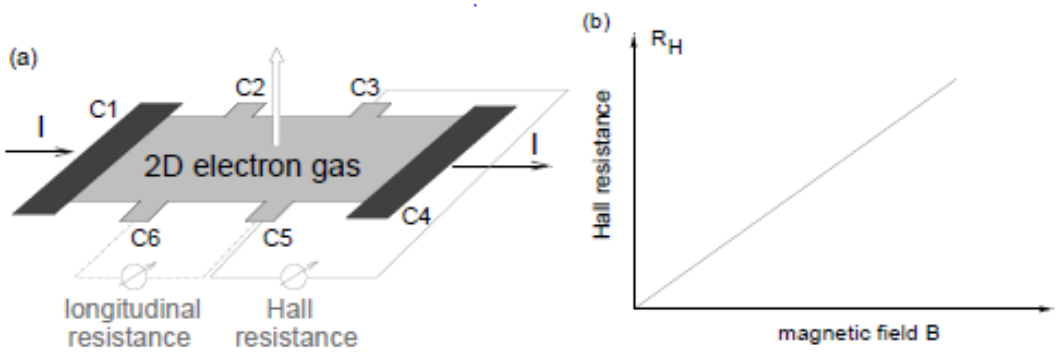

Figure 2: (a) shows how the quantum Hall effect system looks like. 2D electron gas exposed to a magnetic field in the perpendicular direction. The terminals C2, C3 and C5,C6 can be used to measure the longitudinal resistance, while the Hall resistance is measured across the sample between C2,C6 and C3, C5. (b) This graph shows how the Hall resistance is linear to the increasing magnetic field.[1]

then the system of equations becomes

$$\sigma_0 E_x = j_x - \omega_c \tau j_y \tag{22}$$
$$\sigma_0 E_x = -j_x + \omega_c \tau j_y \tag{23}$$

Now one can write this in matrix form

$$\sigma_0 E = \begin{bmatrix} j & \omega_c \tau j \\ -\omega_c \tau j & j \end{bmatrix}$$

and solving for E one get

$$E = \frac{j}{\sigma_0} \begin{bmatrix} 1 & \omega_c \tau \\ -\omega_c \tau & 1 \end{bmatrix}$$

where

$$\rho = \frac{1}{\sigma_0} \begin{bmatrix} 1 & \omega_c \tau \\ -\omega_c \tau & 1 \end{bmatrix}$$

is the resistivity tensor according to the definition $\vec{E} = \rho \vec{j}$. Then one can get the Hall resistivity ρ_H from the off diagonal terms in the resistivity tensor [1, p. 9],

$$\rho_H = \frac{\omega_c \tau}{\sigma_0} = \frac{eB\tau}{m_b \sigma_0} = \frac{B}{n_{el} e}. \tag{24}$$

Finally the Hall resistivity is

$$\rho_H = \frac{B}{n_{el} e}. \tag{25}$$

The goal was to derive the Hall resistance (equation (1)), and the resistivity relates to resistance as follows

$$\rho = R \frac{A}{l}, \tag{26}$$

where A is the cross sectional area and l the length of the sample. However the system is only in two dimensions and the scaling relation between ρ and R is $R \sim \rho L^{2-d}$ where d is the dimension. In this 2D case, the resistance and resistivity is the same from a dimensional point of view and hence the transverse resistance or as it is also called the Hall resistance is [1, p. 10]

$$R_H = \frac{B}{n_{el}e}.$$

This is an interesting result which shows that the Hall resistance does not depend on the particular properties of the material like it's size or geometry, but rather on the charge carrier and the concentration of charges given at an applied magnetic field(see figure 2). This discovery have many applications in a wide range of fields, for example, to measure an external magnetic field one could use a devise based on the Hall effect. One could also use it to determine the sign on the charge carriers. However, this result was derived from the Drude model, which has its restrictions. When one have a 2D system and apply a very strong magnetic field, one get some results which the classical Hall effect does not predict. At a critical magnetic field the longitudinal resistance starts to oscillate as a function of the magnetic field, while the Hall resistance stays linear. This is a phenomena that only can be explained by quantum theories.

2.2 Shubnikov-de Haas effect

In the classical theory it was predicted that the longitudinal resistivity should be independent of the magnetic field. But in 1930 Shubnikov-de Hass effect was discovered, which showed that the longitudinal resistivity or longitudinal resistance did not remain independent, but oscillated as a function of the magnetic field (see figure 3(a)) [1, p. 11]. This is because of something called Landau quantization and is simply the quantization of energy of a 2D electron in a strong magnetic field. If the magnetic field is so strong that the electron goes around a complete orbit without colliding, then the kinetic energy of the electron will be quantized to energy levels called Landau levels, $E_n = \hbar\omega_c(n + 1/2)$, where n is an integer. The radius of the circular path that the electrons travel are called cyclotron radius and it is also quantized [1, p. 12].

To find the critical magnetic field, one has to put the constrains mentioned above. One should assume that the electron has to make an complete orbit without collision, $\omega_c\tau > 1$. Then the critical magnetic field has to be $B_c \simeq m_b/e\tau = \mu^{-1}$, where μ is the mobility [1, p. 12]. To get a better understanding of this, one may use Boltzmann transport equation instead of the Drude model.

From Einsteins relation one can connect the conductivity to the density of states at the Fermi energy [1, p. 12]

$$\sigma_L = e^2 D\rho(E_F). \tag{27}$$

Because of the Landau quantization the density of states become delta peaks (shown in figure 3(b)) given by the Landau levels

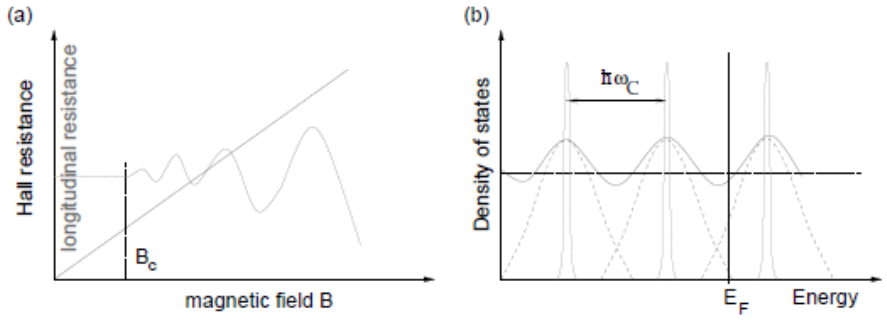

Figure 3: (a) shows how the magnetic field is oscillating as a function of the magnetic field. B_c denotes the critical magnetic field in which the Hall resistance starts to oscillate while the longitudinal resistance stays linear. (b) show the density of states with E_F denoting the Fermi energy. The broadening of the peaks results from the impurities. The horizontal line represents the sum of the overlapping peaks. [1]

$$\rho(E) = \sum_n g_n \delta(E - E_n) \tag{28}$$

where g_n is the degeneracy of the energy levels [1, p. 12].

The samples are never completely clean in reality and thus have some impurities. These impurities causes the peaks to broaden and may even overlap. In such case the density of states oscillate with maximal value at the Fermi energy. As one vary the magnetic field, the distance between the Landau levels changes and the density of states becomes maximal at the Fermi energy and minimal in between two Landau levels. As a result the density of states oscillates as a function of the magnetic field and as a consequence of equation (27), the longitudinal conductivity or resistivity will also oscillate as a function of the magnetic field and explains the Shubnikov-de Hass effect [1, p. 12].

3 Landau Quantization

Before mentioning the integer quantum Hall effect, one has to deal with Landau quantization. As mentioned earlier, Landau quantization is simply the quantization of the kinetic energy of the electron in a 2D system at low temperatures and at high magnetic fields. In this chapter the details of the Landau quantization will be discussed.

3.1 Non relativistic Hamiltonian for a 2D free particle in a zero B-field

In this system one have a two dimensional particle which is translation invariant when there is no magnetic field. The non relativistic Hamiltonian for a free particle then becomes

$$H = \frac{\vec{p}^2}{2m}, \tag{29}$$

where m is the electron mass [1, p. 21]. Since it is translation invariant, the momentum operator commutes with the Hamiltonian. In our case, the electrons are moving around in a metal and one has therefore to consider that the electrons are moving around in a crystal structure, or a lattice where ions are located at every lattice point. In such case one have an electrostatic potential that contributes to the Hamiltonian [1, p.22],

$$H = \frac{\vec{p}^2}{2m} + \sum_i^N V(\vec{r} - \vec{r}_i), \tag{30}$$

where the electrostatic potential from an ion at a lattice site $\vec{r}_i$. Now the Hamiltonian no longer commutes with the momentum operator, hence it is not longer a constant of motion. One can solve this problem by using Bloch's theorem. Bloch's theorem simply states that eigenstates of an electron can be written in terms of a plane wave, $\Psi_{nk}(\vec{r})$ (which can only translate in the direction of the lattice vectors), multiplied by a function, u_{nk}, which has a periodicity of the Bravais lattice.

$$\Psi_{nk}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_{nk}(\vec{r}). \tag{31}$$

The Hamiltonian can be written in terms of momentum, $H(p_x, p_y)$, where $\vec{p}$ is the lattice momentum restricted to the first Brillouin zone. The mass of the Hamiltonian is not the electron mass, but a band mass, which may depend on the direction of motion, since one now describes a quasi-particle [1, p. 23],

$$H = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y}. \tag{32}$$

3.2 Hamiltonian for a non zero B-field

The Hamiltonian for a free electron in a magnetic field is different from the case where there is no magnetic field. To find the Hamiltonian, one has to make a gauge transformation. A gauge transformation is simply a transformation that changes the vector potential without changing the electromagnetic field. In this case the field is the magnetic field and the momentum can be rewritten in its gauge-invariant form in terms of a vector potential,

$$\vec{p} \to \vec{\Pi} = \vec{p} + e\vec{A}(\vec{r}), \tag{33}$$

where the vector potential $\vec{A}$ is related to the magnetic field as $\vec{B} = \vec{\nabla} \times \vec{A}$ [1, p. 26]. To keep $\vec{\Pi}$ gauge-invariant, one can add a gradient term to the momentum and the vector potential. This does not change the magnetic field, since taking the curl of that function will give same result. The transformation of the vector potential can be written as an additional vector field $\vec{\alpha}$ [5, p. 419]

$$\vec{A}(\vec{r}) = \vec{A}(\vec{r}) + \vec{\alpha}. \tag{34}$$

Then the magnetic field in terms of the new vector potential becomes

$$\vec{B} = \nabla \times \vec{A} = \nabla \times (\vec{A} + \vec{\alpha}). \tag{35}$$

Expanding the bracket one get

$$\vec{B} = \vec{B} + \nabla \times \vec{\alpha} \tag{36}$$

and simplifying the expression one end up with

$$\nabla \times \vec{\alpha} = 0. \tag{37}$$

This means that one can define a potential associated with $\vec{\alpha}$

$$\vec{\alpha} = \nabla\lambda, \tag{38}$$

which one can put in the transformation formula equation (34) and one finally get [5, p. 420]

$$\vec{A}(\vec{r}) \rightarrow \ \vec{A}(\vec{r}) + \nabla\lambda(\vec{r}) \tag{39}$$

and a similar approach gives the result for the momentum

$$\vec{p} \rightarrow \vec{p} - e\nabla\lambda(\vec{r}). \tag{40}$$

But the interest lays in describing the electrons in a lattice, which can be problematic. However as long as the lattice constant keep the relation $a \ll l_B$, where $l_B = \sqrt{\hbar/eB}$ is the magnetic length, one can write the gauge in a particular form [1, p. 26], so that the vector potential is,

$$\vec{A}_L(\vec{r}) = B(-y, 0, 0). \tag{41}$$

This gauge is called the Landau gauge, which will be discussed more later on.
So the Hamiltonian transforms as

$$H(\vec{p}) \rightarrow H(\vec{\Pi}) = H(\vec{p} + e\vec{A}) = H^B(\vec{p}, \vec{r}), \tag{42}$$

and the Hamiltonian in the presence of a magnetic field becomes

$$H_S^B = \frac{[\vec{p} + e\vec{A}(\vec{r})]^2}{2m_b}, \tag{43}$$

and for the relativistic case that will be discussed later the Hamiltonian is

$$H_D^B = v[\vec{p} + e\vec{A}(\vec{r})]\vec{\sigma}. \tag{44}$$

3.3 Quantum mechanical interpretation

Now when the Hamiltonians for the particles with and without a magnetic field are known, its time to continue analyzing the system with quantum mechanics. This chapter approaches the system with canonical quantization. One simply wants to describe the theory in a quantum mechanical way. From quantum mechanics one knowns that physical quantities are treated as operators. In classical theory, the momentum is expressed as $\vec{p} = m\vec{v}$, however in quantum mechanics one have the momentum operator instead, $\vec{p} = -i\hbar\frac{\partial}{\partial x}$. One may take the commutator with gauge-invariant momentum

$$[\Pi_x, \Pi_y] = [p_x + eA_x(r), p_y + eA_y(r)] = e([p_x, A_y(x, y)] - [p_y, A_x(x, y)]) =$$

$$= e\Big(\frac{\partial A_x}{\partial x}[p_x, y] + \frac{\partial A_y}{\partial x}[p_x, x] - \frac{\partial A_x}{\partial x}[p_y, x] - \frac{\partial A_x}{\partial y}[p_y, y]\Big). \tag{45}$$

The position and momentum operator does not commute and one have the following commutation relations:

$$[x, P_x] = i\hbar,\ [y, p_y] = i\hbar,\ \text{while}\ \ [x, y] = [p_x, p_y] = [x, p_y] = [y, p_x] = [y, p_x] = 0, \tag{46}$$

so then the momentum commutator (equation (45)) becomes [1, p. 28],

$$[\Pi_x, \Pi_y] = -i\hbar e\Big(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\Big) = -i\hbar e(\nabla \times A)_z = -ie\hbar B, \tag{47}$$

which means that the gauge-invariant momentum does not commute, unlike the momentum as seen above, $[p_x, p_y] = 0$.

The conclusion of this is that one wants to rewrite the Hamiltonian and introduce some ladder operators, which all will be discussed in more detail in the following chapter.

3.4 Non relativistic Landau Levels

In the last chapter it was mentioned that the components of the gauge-invariant operator do not commute. Now one wants to show how the Hamiltonian of the electron in a magnetic field can be reduced to the harmonic oscillator using these commutation relations, since the harmonic oscillator is an approach that is solvable. The Hamiltonian for the harmonic oscillator is [6, p. 42]

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2] \tag{48}$$

What one wants to do now is to factor the Hamiltonian and one should keep in mind that the position and momentum operator do not commute. Hence one get

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \tag{49}$$

and the commutation relation between a_+ and a_- is

$$[a_-, a_+] = \frac{1}{2\hbar m\omega}[ip + m\omega x, -ip + m\omega x] =$$

$$= \frac{1}{2\hbar m\omega}([ip, -ip] + [ip, m\omega x] + [m\omega x, -ip] + [m\omega x, m\omega x]) =$$

$$= \frac{1}{2\hbar m\omega}(im\omega[p, x] - im\omega[x, p]) = \frac{1}{2\hbar m\omega}(2\hbar m\omega) = 1$$

$$\Rightarrow [a_-, a_+] = 1, \tag{50}$$

where $[x, p] = i\hbar$. Now multiply a_- and a_+

$$a_- a_+ = \frac{1}{2\hbar m\omega}(ip + m\omega x)(-ip + m\omega x) = \frac{1}{2\hbar m\omega}[p^2 + (m\omega x)^2] - \frac{i}{2\hbar}[x, p]. \tag{51}$$

The commutation relation between x and p is already known, $[x, p] = i\hbar$, so the final expression becomes,

$$a_- a_+ = \frac{1}{2m}[p^2 + (m\omega x)^2] + \frac{1}{2} = \frac{1}{\hbar\omega}H + \frac{1}{2}. \tag{52}$$

Then to solve for the Hamiltonian one have [6, p. 43],

$$H = \hbar\omega\Big(a_- a_+ - \frac{1}{2}\Big). \tag{53}$$

It does not really matter in which order one writes the ladder operators, as long as one adjust for the sign,

$$H = \hbar\omega\Big(a_\pm a_\mp \pm \frac{1}{2}\Big). \tag{54}$$

If one then look at the Schrödinger equation, $H\Psi = E\Psi$, one can see how this ladder operator acts on the energy. Lets multiply Ψ with a_+ and see what it does.

$$\begin{aligned} H(a_+\Psi) &= \hbar\omega\Big(a_+ a_- + \frac{1}{2}\Big)(a_+\Psi) = \hbar\omega\Big(a_+ a_- a_+ + \frac{1}{2}a_+\Big)\Psi = \\ &= \hbar\omega a_+\Big(a_- a_+ + \frac{1}{2}\Big)\Psi = a_+\Big(\hbar\omega\Big(a_+ a_- + 1 + \frac{1}{2}\Big)\Psi\Big) = \\ &= \hbar\omega a_+(H + \hbar\omega)\Psi = a_+(E + \hbar\omega)\Psi = (E + \hbar\omega)(a_+\Psi). \end{aligned} \tag{55}$$

So by acting with a_+ increases the energy to a next energy level and the same would be true if one would make the same calculation for a_-, but it would lower one step instead [6, p. 42]. The conclusion is that one have discrete energy levels. To find the energy level one can just use the Schrödinger equation again. However first one must set up boundaries to the ladder operators. It is not realistic to just lower the energy states forever, so if a_- acts on the lowest state, it annihilates it by definition.
The key is to find the lowest state, then one can just act with a_+ to find all the other states. The lowest state can be found with the help of Schrödinger equation

$$\hbar\omega\Big(a_+ a_- + \frac{1}{2}\Big)\Psi_0 = E_0\Psi_0 \tag{56}$$

and by definition $a_-\Psi_0 = 0$, so then one gets

$$E_0 = \frac{1}{2}\hbar\omega \tag{57}$$

as the lowest energy level. Then one can just apply a_+ operator n number of times to get the nth energy level. So the energy levels for the harmonic oscillator is

$$E_n = \hbar\omega\Big(n + \frac{1}{2}\Big). \tag{58}$$

If one now go back to the Hamiltonian for a 2D electron in a magnetic field where one wants to express the Hamiltonian in terms of the ladder operators. The non relativistic Hamiltonian can be written as equation (43),

$$H_S^B = \frac{1}{2m_b}(\Pi_x^2 + \Pi_y^2). \tag{59}$$

The ladder operators in this case is defined, in terms of the gauge-invariant operators as

$$a = \frac{l_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y),\ a^\dagger = \frac{l_B}{\sqrt{2}\hbar}(\Pi_x + i\Pi_y). \tag{60}$$

If one would commute a and $a^\dagger$ one would get,

$$[a, a^\dagger] = \frac{l_B^2}{2\hbar^2}\Big[\Pi_x - i\Pi_y, \Pi_x + i\Pi_y\Big] =$$

$$= \frac{l_B^2}{2\hbar^2}\Big(\Big[\Pi_x, \Pi_x\Big] + \Big[\Pi_x, i\Pi_y\Big] + \Big[-i\Pi_y, \Pi_x\Big] + \Big[-i\Pi_y, i\Pi_y\Big]\Big) =$$

$$= \frac{l_B^2}{2\hbar^2}\Big(\frac{2\hbar^2}{l_B^2}\Big) = 1$$

$$\Rightarrow [a, a^\dagger] = 1 \tag{61}$$

and one can see that this is exactly the same result as in the case of the harmonic oscillator. Expressing the Hamiltonian in terms of these ladder operators one get

$$H_S^B = \frac{\hbar^2}{4ml_B^2}[a^{\dagger 2} + a^\dagger a + aa^\dagger + a^2 - (a^{\dagger 2} - a^\dagger a - aa^\dagger + a^2)] =$$

$$= \frac{\hbar^2}{2ml_B^2}(a^\dagger a + aa^\dagger) = \frac{\hbar^2}{ml_B^2}\Big(a^\dagger a + \frac{1}{2}\Big) =$$

$$= \hbar\omega_c\Big(a^\dagger a + \frac{1}{2}\Big), \tag{62}$$

$$\Rightarrow H = \hbar\omega_c\Big(a^\dagger a + \frac{1}{2}\Big). \tag{63}$$

Now the Hamiltonian in terms of the ladder operators are known. Just like the way done previously, one wants to find the energy levels. This is done as before, by first finding the lowest state and then one apply $a^\dagger$ to the lowest state n number of times.
From the Schrödinger equation the lowest energy state is

$$\hbar\omega_c\Big(a^\dagger a + \frac{1}{2}\Big)\Psi_0 = E_0\Psi_0, \tag{64}$$

where the lowest state gets annihilated ($a\Psi_0 = 0$),

$$E_0 = \hbar\omega_c\frac{1}{2}. \tag{65}$$

Then by applying $a^\dagger$ to climb up to the higher energy levels, one get the Landau levels [1, p.30],

$$E = \hbar\omega_c\Big(n + \frac{1}{2}\Big). \tag{66}$$

which is plotted against the magnetic field in figure 4(a) for different values of n and $\omega_c = \hbar^2/ml_B$ is the cyclotron frequency.

3.5 Relativistic Landau levels

In this chapter one treats the Hamiltonian in the relativistic case. As before one write the Hamiltonian in terms of of ladder operators,

$$H_D^B = v \begin{pmatrix} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{pmatrix} = \sqrt{2}\frac{\hbar v}{l_B} \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} = \hbar\omega' \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \tag{67}$$

where the corresponding cyclotron frequency is $\omega' = \sqrt{2}v/l_B$ in the relativistic case. The 2x2 Hamiltonian matrix now comes from the Dirac theory. The band mass is zero in graphene, so one cannot write the frequency in the usual way $\omega = eB/m_b$ [1, p. 31].

To find the eigenvalues to the Hamiltonian one just solves Schrödinger's equation $H_D^B\psi_n = \epsilon_n\psi_n$. Then one will find that the eigenstates are 2-spinor eigenstates

$$\psi_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{68}$$

So one needs to solve two equations for each u_n and v_n,

$$\hbar\omega' a^\dagger v_n = \epsilon_n u_n \tag{69}$$
$$\hbar\omega' a^\dagger u_n = \epsilon_n v_n \tag{70}$$

and one will find that

$$u_n = \frac{\hbar\omega' a v_n}{\epsilon_n} \tag{71}$$
$$u_n = \frac{\epsilon_n v_n}{\hbar\omega' a^\dagger} \tag{72}$$

$$\Rightarrow \frac{\hbar\omega' a v_n}{\epsilon_n} = \frac{\epsilon_n v_n}{\hbar\omega' a^\dagger} \tag{73}$$

$$\Rightarrow a^\dagger a v_n = \left(\frac{\epsilon_n}{\hbar\omega'}\right)^2 v_n. \tag{74}$$

One can then relate $a^\dagger a = n$ as seen when the energy levels for the non relativistic case was derived. Then one will find that the energy and this eigenstate is related as

$$\epsilon_n^2 = (\hbar\omega')^2 n \tag{75}$$

which of course has two solutions,

$$\epsilon_{\lambda,n} = \frac{\lambda\hbar v}{l_B}\sqrt{2n}, \tag{76}$$

where $\lambda = \pm$ is a quantum number. $\epsilon_{\lambda,n}$ is plotted in figure 4(b) where one can see two solutions, one of which show negative energies. This introduces the theory of holes, where one have to consider both electrons and holes.

The spinors then finally becomes [1, p. 32]

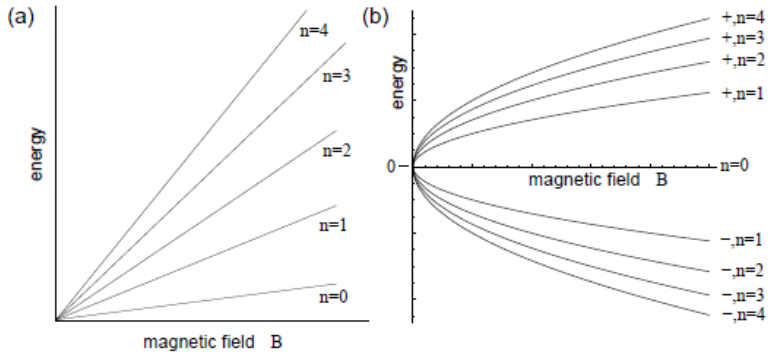

Figure 4: (a) shows the non relativistic case when energy is a linear function of the magnetic field, plotted for different values of n. (b) show the relativistic case for the two solutions. Also plotted for different values of n. [1]

$$\psi_{n=0} = \begin{pmatrix} 0 \\ |n=0\rangle \end{pmatrix} \tag{77}$$

for the case $n = 0$ and

$$\psi_{n\neq 0} = \begin{pmatrix} |n-1\rangle \\ \lambda|n\rangle \end{pmatrix} \tag{78}$$

for the case when $n \neq 0$.

3.6 Level degeneracy

The Hamiltonian for non relativistic particles only depends on two pairs of conjugate operators (x, p_x, y, p_y). But when one express the Hamiltonian in terms of the gauge-invariant operator or the ladder operators, it only depends on one pair of conjugate operators. This means that one has to look for a second pair of conjugate operators that commute with the Hamiltonian such that it gives rise to the level degeneracy of the Landau levels.

Consider the pseudo-momentum operator [1, p. 34]

$$\tilde{\Pi} = \vec{p} - e\vec{A}(\vec{r}) \tag{79}$$

so that one can express the momentum in terms of this pseudo-momentum and the gauge-invariant momentum defined previously,

$$\vec{p} = \frac{1}{2}(\Pi + \tilde{\Pi}) \tag{80}$$

and also the vector potential as

$$\vec{A}(\vec{r}) = \frac{1}{2}(\Pi - \tilde{\Pi}). \tag{81}$$

The gauge-invariant operator and the pseudo-momentum operator does not describe something physical, because they depend on the gauge that you are using.

But if one try to commute the components of the pseudo-momentum operator, one get

$$\Big[\tilde{\Pi}_x, \tilde{\Pi}_y\Big] = \Big[p_x - eA_x(r), p_y - eA_y(r)\Big] = -\Big[p_x, eA_y(r)\Big] + \Big[p_y, eA_y(r)\Big] =$$

$$= e\Big(-\frac{\partial A}{\partial x}[p_x, x] - \frac{\partial A}{\partial y}[p_x, y] + \frac{\partial A}{\partial x}[p_y, x] + \frac{\partial A}{\partial y}[p_y, y]\Big) =$$

$$e\Big(i\hbar\frac{\partial A}{\partial x} - i\hbar\frac{\partial A}{\partial y}\Big) = i\hbar e\Big(\frac{\partial A}{\partial x} - \frac{\partial A}{\partial y}\Big) =$$

$$= -ie\hbar(\nabla \times \vec{A})_z = -i\frac{\hbar^2}{l_B^2}$$

$$\implies \Big[\tilde{\Pi}_x, \tilde{\Pi}_y\Big] = -i\frac{\hbar^2}{l_B^2}. \tag{82}$$

So the commutators are gauge-invariant [1, p. 34]. Other commutation relations can be found in a similar way,

$$\Big[\Pi_x, \tilde{\Pi}_x\Big] = 2ie\hbar\frac{\partial A_x}{\partial x}, \tag{83}$$

$$\Big[\Pi_y, \tilde{\Pi}_y\Big] = 2ie\hbar\frac{\partial A_y}{\partial y}, \tag{84}$$

$$\Big[\Pi_x, \tilde{\Pi}_y\Big] = ie\hbar\Big(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\Big) = -\Big[\tilde{\Pi}_x, \Pi_y\Big]. \tag{85}$$

In order to commute with the Hamiltonian one need these terms (83), (84) and (85) to vanish. This can be done by finding a particular gauge such that they do vanish. There are two different gauges in this case, the symmetric gauge and the Landau gauge [1, p. 35].

$$\text{Symmetric gauge: } A_S(\vec{r}) = \frac{B}{2}(-y, x, 0) \tag{86}$$

$$\text{Landau gauge: } A_L(\vec{r}) = B(-y, 0, 0). \tag{87}$$

The use of these different gauges has different advantages. The symmetric gauge has the advantage that, as will be discussed below, it is easier to use the semi-classical approach than the Landau gauge. But the Landau gauge is more practical when it comes to geometries with the translation invariance in the y-direction, since the Landau gaguge is translation invariant in the x-direction.

One can introduce ladder operators corresponding to the pseudo-momentum operator and they are defined as [1, p. 35]

$$b = \frac{l_B}{\sqrt{2}\hbar}(\tilde{\Pi}_x + i\tilde{\Pi}_y)$$

$$b^\dagger = \frac{l_B}{\sqrt{2}\hbar}(\tilde{\Pi}_x - i\tilde{\Pi}_y) \tag{88}$$

which satisfy

$$[b, a^\dagger] = \Big[\frac{l_B}{\sqrt{2}\hbar}(\tilde{\Pi}_x + i\tilde{\Pi}_y), \frac{l_B}{\sqrt{2}\hbar}(\Pi_x + i\Pi_y)\Big] =$$

$$= \frac{l_B^2}{2\hbar^2}\Big(\Big[\tilde{\Pi}_x, \Pi_x\Big] + \Big[\tilde{\Pi}_x, i\Pi_y\Big] + \Big[i\tilde{\Pi}_y, \Pi_x\Big] + \Big[i\tilde{\Pi}_y, i\Pi_y\Big]\Big)$$

$$\implies [b, a^\dagger] = 0. \tag{89}$$

The same is true for the Hamiltonian,

$$\implies [b^\dagger, H_B] = 0. \tag{90}$$

Then one can set up an eigenvalue equation, which will introduce a new quantum number, $m \geq 0$,

$$b^\dagger b|m\rangle = m|m\rangle. \tag{91}$$

3.7 Semi-classical interpretation of the level degeneracy

The introduction of the symmetry gauge was done in the previous chapter and it was concluded that it was a good choice as a gauge for the semi-classical approach. In this chapter the semi-classical interpretation of the level degeneracy will be introduced with the help of the symmetric gauge to make a physical representation of the pseudo-momentum operator. As the name suggests, the semi-classical approach is partially describing the system in a classical manner and then also introduce a quantum mechanical treatment.
If one starts with describing an electron as a classical object moving in a magnetic field, one gets the equation of motion from Newton's second law [1, p. 37],

$$m_b\ddot{r} = -e(\dot{r} \times \vec{B}), \tag{92}$$

or written in the two different components x and y (since it is a particle moving in two dimensions),

$$\ddot{x} = -\omega_c\dot{y} \tag{93}$$

$$\ddot{y} = \omega_c\dot{x}, \tag{94}$$

where $\omega_c = eB/m_b$ is the cyclotron frequency. The solution to these equations generates a circular orbit with radius $\vec{R}$ (see figure 5),

$$x(t) = X - rsin(\omega_c t + \phi) \tag{95}$$

$$y(t) = Y - rcos(\omega_c t + \phi). \tag{96}$$

The radius $\vec{R} = (X, Y)$ is called the guiding center. As the particle is exposed to a magnetic field, it will go in orbit around a center point, which is the guiding center. When the particle is in a drift motion, the patten will look like a helix, orbiting around the axis of the guiding center . So now one needs to connect the

guiding center to the pseudo-momentum. This can be done using the symmetric gauge in terms of pseudo-momentum,

$$eA_s(r) = \frac{1}{2}(\Pi - \tilde{\Pi}). \tag{97}$$

Then one get

$$y = \frac{\tilde{\Pi}_x}{eB} - \frac{\Pi_x}{eB} \tag{98}$$

$$x = \frac{\tilde{\Pi}_y}{eB} - \frac{\Pi_y}{eB}. \tag{99}$$

By integrating (92) one also knows that

$$y = Y - \frac{\Pi_x}{eB} \tag{100}$$

$$x = X + \frac{\Pi_y}{eB} \tag{101}$$

and by this one can relate the guiding center to the pseudo-momentum

$$X = -\frac{\tilde{\Pi}_y}{eB} \tag{102}$$

$$Y = \frac{\tilde{\Pi}_x}{eB}. \tag{103}$$

From this one can conclude that the pseudo-momentum are constants of motion in terms of the guiding center. One can therefore expect the operators to commute with the Hamiltonian. One interesting aspect is that one know the commutation relation between the pseudo-momentum components $[\tilde{\Pi}_x, \tilde{\Pi}_y] = i\hbar^2/l_B^2$ and therefore also the commutation relation between the guiding center components,

$$[X, Y] = il_B^2. \tag{104}$$

This results in the Heisenberg's uncertany relation implying

$$\Delta X \Delta Y = 2\pi l_B^2. \tag{105}$$

The guiding center cannot be determined exactly, but is distributed over a surface A represented by the grey area in figure 5,

$$N_B = \frac{A}{\Delta X \Delta Y} = \frac{A}{2\pi l_B^2} = n_B \times A \tag{106}$$

where n_B is the flux density,

$$n_B = \frac{1}{2\pi l_B^2} = \frac{B}{h/e}. \tag{107}$$

The conclusion of this is that "the number of quantum states in a Landau level equals the number of flux quanta threading the sample surface A and each Landau level is macroscopically degenerate" [1, p. 38].

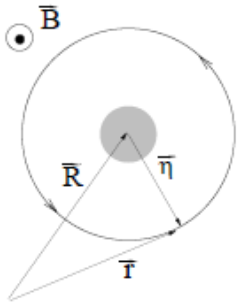

Figure 5: This figure shows the cyclotron motion of an electron exposed to a perpendicular magnetic field coming out of the page. The grey area represents the guiding center "cloud". The reason for the cloud is that the components of the guiding center operator do not commute. [1]

Another important thing that needs to be defined is the filling factor. Since electrons are fermions, they obey the Pauli exclusion principle and therefore cannot occupy same quantum state. The electrons will at first start to occupy the lowest Landau levels, but as they are filled up, higher energy levels will be occupied. Therefore one may define the filling factor as the ratio between the number of electrons and the flux quanta [1, p. 39],

$$\nu = \frac{N_{el}}{N_B} = \frac{n_{el}}{n_B} = \frac{hn_{el}}{eB}. \tag{108}$$

3.8 Eigenstates and the wave function in the symmetric and Landau gauge

In this chapter one wants to define some wave functions in the different gauges with some quantum mechanics starting with the symmetric gauge. The approach here is that one wants to find the wave function by using the differential equation instead of ladder operators. As already known, one can determined all quantum states from the ladder operators, which in this case is [1, p. 39],

$$|n, m\rangle = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!}\sqrt{m!}} |n = 0, m = 0\rangle. \tag{109}$$

Now equation(109) translates to differential equation with the wave function $\phi_{n,m}(x, y)$, but in order to do that, one has to use the following equations which was discussed previously

$$a = \frac{l_B}{\sqrt{2}\hbar}(\Pi_x - i\Pi_y)$$

$$\Pi = \vec{p} + e\vec{A}(\vec{r}),$$

where $\vec{p} = -i\hbar\nabla$ and $\nabla = (\partial_x, \partial_y)$. After inserting this into a one get,

$$a = -i\sqrt{2}\Big[\frac{l_B}{2}(\partial_x - i\partial_y) + \frac{x - iy}{4l_B}\Big]. \tag{110}$$

In our quest to find the wave function one may introduce complex coordinates and transfer equation(95) into them. Since it is in two dimensions the complex coordinates becomes,

$$z = x - iy \ , \ z^* = x + iy \tag{111}$$

$$\partial = \frac{\partial_x + i\partial_y}{2} \ , \ \overline{\partial} = \frac{\partial_x - i\partial_y}{2}. \tag{112}$$

Now plug that into equation (109)

$$\Big(\frac{z}{4l_B} + l_B\overline{\partial}\Big)\phi_{n=0}(z, z^*) = 0, \tag{113}$$

which is now expressed as a differential equation in the lowest Landau level. The solution to the wave function yields

$$\phi_{n=0}(z, z^*) = f(z)e^{-|z|^2/4l_B^2} \tag{114}$$

where $f(z)$ is an arbitrarily analytic function. As a consequence of that $f(z)$ brings in extra degrees of freedom and therefor can be related to the second quantum number m. To deal with m one makes a similar approach as previously discussed in the case of n. First introduction the ladder operators,

$$\begin{aligned} a &= -i\sqrt{2}\Big(\frac{z}{4l_B} + l_B\overline{\partial}\Big), \ a^\dagger = i\sqrt{2}\Big(\frac{z^*}{4l_B} - l_B\partial\Big) \\ b &= -i\sqrt{2}\Big(\frac{z^*}{4l_B} + l_B\partial\Big), \ b^\dagger = i\sqrt{2}\Big(\frac{z}{4l_B} - l_B\overline{\partial}\Big) \end{aligned} \tag{115}$$

and from this one get the differential equation [1, p. 40]

$$(z^* + 4l_B^2\partial)\phi'_{m=0}(z, z^*) = 0 \tag{116}$$

which yields the solution

$$\phi'_{m=0}(z, z^*) = g(z^*)e^{|z|^2/4l_B^2}. \tag{117}$$

$g(z^*)$ is anti-analytic and therefor $\partial g(z^*) = 0$. The wave function then becomes Gaussian with a constant factor to normalize,

$$\phi_{n=0,m=0}(z, z^*) = \langle z, z^*|n = 0, m = 0\rangle = \frac{1}{\sqrt{2\pi l_B^2}}e^{-|z|^2/4l_B^2}, \tag{118}$$

since it is both analytic and anti-analytic. To achieve the lowest Landau level, one can just use equation (94),

$$\phi_{n=0,m}(z, z^*) = \frac{i^m}{\sqrt{2\pi l_B^2 m!}}\Big(\frac{z}{\sqrt{2}l_B}\Big)^m e^{-|z|^2/4l_B^2}. \tag{119}$$

Last thing covered about the symmetric gauge is that one may find an average value of the guiding center. To do this, one starts with the guiding center in the state $|n = 0, m\rangle$ and then write the guiding center in terms of ladder operators and work the way from there.

$$X = \frac{l_B}{i\sqrt{2}}(b^\dagger - b)$$

$$Y = \frac{l_B}{\sqrt{2}}(b^\dagger + b). \tag{120}$$

Then one will find that the average value is,

$$\langle \vec{R} \rangle = \langle n = 0 | \vec{R} | n = 0, m \rangle = 0 \tag{121}$$

but if one take the absolute value

$$\langle |\vec{R}| \rangle = \langle \sqrt{X^2 + Y^2} \rangle = \Big\langle \sqrt{\frac{-l_B^2}{2}(b^\dagger - b)^2 + \frac{l_B^2}{2}(b^\dagger + b)^2} \Big\rangle =$$

$$= l_B \Big\langle \sqrt{\frac{-1}{2}((b^\dagger)^2 - b^\dagger b - bb^\dagger + b^2) + \frac{1}{2}((b^\dagger)^2 + b^\dagger b + bb^\dagger + b^2)} \Big\rangle =$$

$$= l_B \langle \sqrt{b^\dagger b + bb^\dagger} \rangle = l_B \langle \sqrt{2b^\dagger b + 1} \rangle = l_B \sqrt{2m + 1} \tag{122}$$

one will find that that the quantum states are located on a circle with radius $\langle |\vec{R}| \rangle = l_B \sqrt{2m + 1}$. Here the ladder operators are defined as usual,

$$b^\dagger |n, m\rangle = \sqrt{m + 1} |n, m + 1\rangle, \; b|n, m\rangle = \sqrt{m} |n, m - 1\rangle. \tag{123}$$

The maximum number of quantum states one can fit within the circle is

$$M = \frac{A}{2\pi l_B^2} = n_b \times A = N_B, \tag{124}$$

where A is the surface contained in the circle with maximum radius $R_{max} = \pi l_B^2 (2M + 1)$ for maximum value of quantum state m.

This is what is covered about the symmetric gauge, now over to the Landau gauge. When the sample has a rectangular scape, it is better to use the Landau gauge, since it is only x-depenant. Then one can make a plane-wave anatz to the wave function,

$$\psi_{n,k}(x, y) = \frac{e^{ikx}}{\sqrt{L}} \chi_{n,k}(y), \tag{125}$$

where L is the length of the sample. Then the Hamiltonian becomes,

$$H_S^B = \frac{(p_x - eBy)^2}{2m} + \frac{p_y^2}{2m} = \frac{p_y^2}{2m} + \frac{1}{2} m \omega_c (y - y_0)^2 \tag{126}$$

where $y_0 = k l_B^2$. This Hamilitonian is just the Hamiltonian for an oscillator with eigenstates,

$$\chi_{n,k}(y) = H_n\Big(\frac{y - y_0}{l_B}\Big) e^{-(y - y_0)^2 / 4 l_B^2}, \tag{127}$$

and H_n is the Hermite polynomial. y_0 which is the point where the oscillating motion oscillates about, corresponds to Y, the guiding center component. Since the

guiding center components do not commute, X cannot be determined precisely, but is located somewhere along the sample length L.

To find the number of quantum states located within the rectangular surface with dimensions L and W, one may use periodic boundary conditions, $k = M \times 2\pi/L$ in the x-direction and then use the same approach as in the case of the symmetric gauge. So the number of states in the rectangular surface turns out to be

$$M = N_B = n_B \times LW = n_B \times A \tag{128}$$

where y varies between $y_{min} = 0$ and $y_{max} = W$, in the y-direction.

4 Integer Quantum Hall effect

Now when the Landau quantization has been established, one can start to discuss integer quantum Hall effect. Some of the things that will be discussed in this chapter is the confinement potential, which plays an important role. Quantization of the kinetic energy of a two-dimensional particle discussed with the Landau quantization, has to be related to the quantization of the resistance. The background of the plateau pattern will also be covered.

4.1 Electronic motion in an external electrostatic potential

When considering a current moving in a sample, one has to consider the particles in that sample to effect the charge carriers in the current. This is described by two kinds of potentials, confinement potential and an impurity potential. Since the current is confined to the sample, the confinement potential varies only along the y-direction, ($V_{conf}(y)$) and does not effect the particles moving along the sample in the x-direction. The other potential that need to be considered is the impurity potential caused by the impurities in the sample, $V_{imp}(x, y)$, which effects the particles in both directions. If one wants to describe a particle moving in this system, one has to use the total potential in the Hamiltonian which is the sum of these potentials [1, p. 45],

$$V(\vec{r}) = V_{conf}(y) + V_{imp}(x, y). \tag{129}$$

4.2 Semi-classical approach

In the presences of the potential $V(\vec{r})$ the Hamiltonian will be effected in such away that it does not commute with R any longer. The consequence of this is that $V(\vec{r})$ lifts the Landau degeneracy, which means that as the system is exposed to the external field, the degeneracy is reduced since the energy levels are split. The guiding center is translation invariant but the electrostatic potential, $V(\vec{r})$ breaks this invariance.

When $V(\vec{r})$ is smooth on the length scale of l_B it does not generate Landau mixing, one can approximate the argument of the potential with the position of the guiding center, $V(\vec{r}) \simeq V(R)$. Since $V(\vec{R})$ and $\vec{R}$ do not commute, one may consider the Heisenberg equation of motion [4],

$$\frac{dA}{dt} = \frac{1}{i\hbar}\Big[A, H\Big],$$

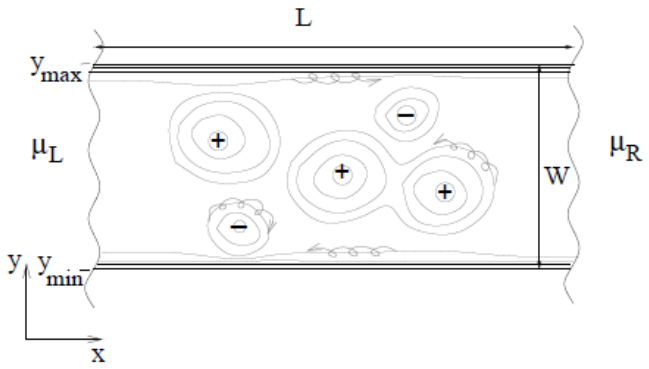

Figure 6: This figure shows the potential landscape for the charge carriers. The circular lines and the thin lines at the edge represents the equipotential lines where the closed lines are localized states and the open lines are the extended states. The charge carriers are confined in the region between y_{min} and y_{max}. μ_L and μ_R are the chemical potential at each contact and the dimensions of the sample is assumed to be $L \gg W \gg \xi \gg l_B$, where W is the width and L the length and ξ the length scale at which the electrostatic potential varies. [1]

which in this case can be written as

$$i\hbar\dot{X} = [X, H] = [X, V(\vec{R})] = \frac{\partial V}{\partial Y}[X, Y] = il_B^2 \frac{\partial V}{\partial Y}$$

$$i\hbar\dot{Y} = [Y, V(\vec{R})] = -il_B^2 \frac{\partial V}{\partial X}. \tag{130}$$

The guiding center component is moving along the equipotential lines since $\vec{R}$ is perpendicular to $V(\vec{R})$. This corresponds to the Hall drift,

$$\langle \dot{R} \rangle = \frac{-\nabla V \times \vec{B}}{eB^2} = \frac{\vec{E} \times \vec{B}}{B^2} = \vec{v}_D. \tag{131}$$

Due to the potential $V(\vec{R})$ a landscape is formed with hills and valleys which the charge carriers are moving through. The impurity in the sample changes the landscape in such way that hills or a valleys are formed, depending on the charge of the particle. These impurities causes the equipotential lines to close on them selfs and this will cause the charge carriers to orbit the bulk/valley, counter-clockwise or clockwise depending on the charge. This means that the charge carriers are localized around this topological region and therefore may not contribute to the actual current in the sample. One therefore call these charge carriers localized. Not only can a very strong magnetic field localize the charge carriers in closed orbits, but so does the impurities in the sample. At the edge, the equipotential lines are not closed, but open because the confinement potential $V_{conf}(y)$ increases at the edges such that it confines the charge carriers to the sample. These states are then called extended states and contribute to a current in the sample [1, p. 46].

4.3 Electrostatic potential with translation invariance in the x-direction

The assumptions made in the previous chapter has some constraints, since one assumed that the confinement potential was smooth on a scale of l_B, while confinement potential varies significantly on the scale of l_B at the edges (y_{min}, y_{max}) and therefore one needs to treat it accurately while solving the Schrödinger equation. One choose to work within the Landau gauge, since it is translation invariant in x-direction. In this case the Hamiltonian becomes,

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c(y - y_0)^2 + V_{conf}(y), \tag{132}$$

where $y_0 = kl_B^2$ is the center of oscillation and k is the wave vector in the x-direction. Then one can expand the potential around this point,

$$V_{conf}(y) = V(y_0) + \frac{\partial V(y_0)}{\partial y}(y - y_0) + \mathcal{O}\Big(\frac{\partial^2 V}{\partial y^2}\Big) =$$

$$= V(y_0) - eE(y_0)(y - y_0) + \mathcal{O}\Big(\frac{\partial^2 V}{\partial y^2}\Big). \tag{133}$$

and writing the Hamiltonian in terms of the expanded confinement potential and excluding the second and higher terms one gets [1, p. 46],

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c(y - y_0')^2 + V_{conf}(y_0') \tag{134}$$

where the harmonic oscillator is now shifted $y_0 \to y_0' = y_0 + eE(y_0)/m\omega_c^2$. The energy levels given by this Hamiltonian are,

$$\epsilon_{n,y_0'} = \hbar\omega_c\Big(n + \frac{1}{2}\Big) + V(y_0') \tag{135}$$

which gives the same energy levels as before, but with an additional term that solves the problem of large variation at the edges.

4.4 Landau levels and conductance

In this chapter one wants to calculate the conductance of a filled Landau level. Consider all Landau levels up to the nth level to be occupied, then using the Landau gauge one will find that the current going throw the nth Landau level in the sample to be [1, p. 48],

$$I_n^x = -\frac{e}{L}\sum_k \langle n, k|v_x|n, k\rangle, \tag{136}$$

where the wave vector $k = 2\pi m/L$ and the current is only x-dependent.

The velocity v_x can with the help of Heisenberg's equation of motion be written as

$$i\hbar v_x = [v_x, H] = \frac{\partial H}{\partial p_x}[x, p_x] = \frac{1}{\hbar}\frac{\partial H}{\partial k}[x, p_x] =$$

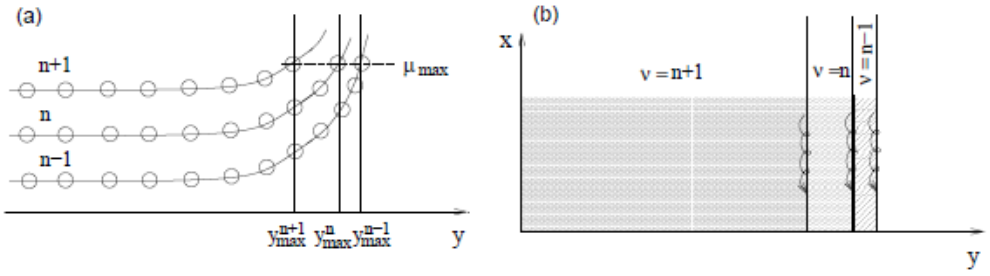

Figure 7: (a) show how the energy levels are bent upwards as one moves towards the edge. (b) shows the filling factor jump that corresponds to the Hall drift and chirality. [1]

$$= \frac{i\hbar}{\hbar}\frac{\partial H}{\partial k} = i\frac{\partial H}{\partial k} = \frac{1}{\hbar}\frac{\partial \epsilon}{\partial k}$$

and for the state $|n,k\rangle$

$$\Rightarrow \langle n,k|v_x|n,k\rangle = \frac{1}{\hbar}\frac{\partial \epsilon_{n,k}}{\partial k} = \frac{L}{2\pi\hbar}\frac{\Delta\epsilon_{n,m}}{\Delta m}. \tag{137}$$

Solving for $\Delta m = 1$ one get

$$\langle n,k|v_x|n,k\rangle = \frac{L}{h}(\epsilon_{n,m+1} - \epsilon_{n,m}) \tag{138}$$

and inserting this into the equation (136) one get

$$I_n = -\frac{e}{L}\sum_m \frac{L}{h}(\epsilon_{n,m+1} - \epsilon_{n,m}). \tag{139}$$

All the terms in the sum cancel except for the edge terms

$$I_n = -\frac{e}{h}(\epsilon_{n,m_{min}} - \epsilon_{n,m_{max}}) = -\frac{e}{h}(\mu_{min} - \mu_{max}), \tag{140}$$

where μ_{min} and μ_{max} are the chemical potential corresponding to these energies at the edges (see figure 6). The difference between these potentials corresponds to the Hall voltage V across the sample,

$$I_n = -\frac{e}{h}(\mu_{min} - \mu_{max}) = \frac{e^2}{h}V. \tag{141}$$

Then one can obtain the conductance in the nth Landau level, which is now quantized,

$$G = \sum_{n'=0}^{n-1} G_{n'} = n\frac{e^2}{h}. \tag{142}$$

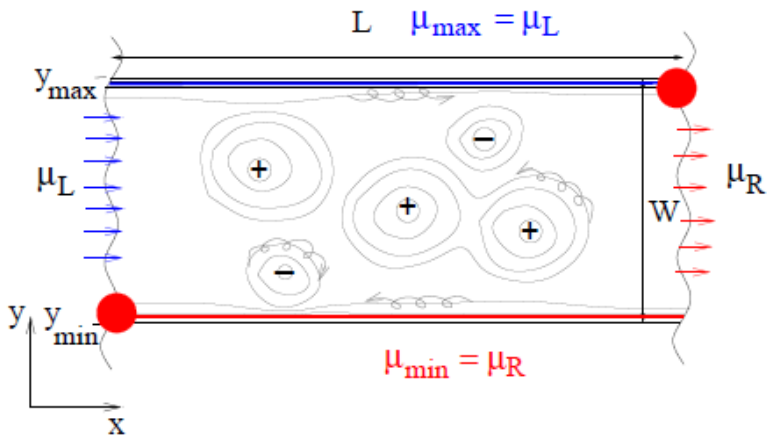

Figure 8: Here one can see the potential landscape and the left contact μ_L is in thermodynamic equilibrium with the upper edge μ_{max}, while the right contact μ_R is in thermodynamical equilibrium with lower edge μ_{min}. The red dots corresponds to the hotspots, where the chemical potential suddenly drops as it gets in contact with the other side. The Hall voltage is the voltage between these edges and the resistance between the contacts are the Hall resistance. [1]

4.5 Edge states

What one wants to describe here is how the current is transported at the edges of the sample. The confinement potential increases sharply at the edge which introduces a perpendicular motion, because if the current is in the state n and the confinement potential increases such that the state n reaches the maximal value μ_{max}, it will jump down towards the lower level $n-1$ due to this bent(see figure 7). This is explained by the Hall drift and is called chirality (see figure 7). The chirality is constant on the edges and thus corresponds to no directional changes in the current. The chirality is in the opposite direction at the opposite edge and as long as electrons cannot jump between these two, they cannot backscatter [1, p. 50]. Since the edges are separated by a macroscopic distance, the probability for that to happen is very low. Electrons moving between different states on the same chirality will not change its direction.

4.6 Integer quantum Hall effect and percolation

In this chapter one wants to explain why a plateau pattern is shown when measuring the Hall resistance against the magnetic field. The approach that one needs to take is the semi-classical localization of charge carriers. Consider that initially one have n filled Landau levels, where the nth Landau level is unoccupied, then one will measure a zero longitudinal resistance and a Hall resistance of $R_H = h/e^2n$. If one then imagine a topological landscape with hills and valleys that the electrons or holes are traveling through, then as the magnetic field decreases, the nth Landau level start to fill up in a valley in the potential landscape. Because the filling factor is defined as equation (108). The electrons start to fill up a valley and are therefore in a localized state and do not contribute to any current and the Hall resistance remain constant (see figure 9).

If one decreases the magnetic field even more, the valley of occupied states becomes larger and may in fact become so large that it overwhelms the valley and

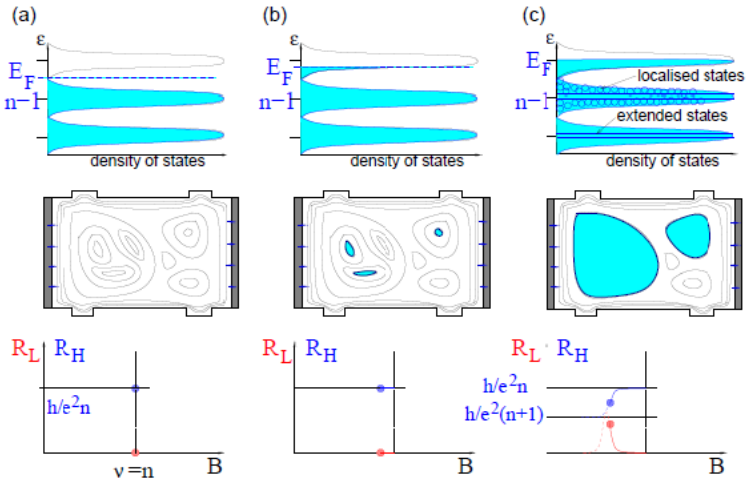

Figure 9: At the top row is the density of states shown with the Fermi energy E_F placed out. The middle row shows the potential landscape with the equipotential lines. The lowest row show the Hall and longitudial resistance plotted against the magnetic field. One can see how the valleys in the closed lines starts to get filled up and grow as the filling factor increases. [1]

connect the two opposite edges to each other. This enables the electrons to travel from the upper edge to the lower edge and since the chirality is the opposite on the other edge, the electrons backscatter. This leak of electrons will cause the voltage to decrease between the left and right contacts and cause the longitudinal resistance to be non zero. The Hall resistance will not be quantized anymore and undergo a plateau transition, jump to the next plateau. This is the reason why there is a plateau pattern when one measures the Hall resistance against the magnetic field. Same arguments can be applied for holes, but they will see the potential landscape in the opposite way, since they have opposite charge.

One may also note that the plateau transitions are second-order quantum phase transitions that is described by universal laws. At some critical magnetic field the phase transition occurs,

$$\varepsilon \sim |B - B_c|^{-\nu} \tag{143}$$

where ν is a critical exponent.

5 Relativistic Quantum Hall Effect in Graphene

Relativistic quantum Hall effect in graphene will be discussed in this chapter, where subjects like confinement potential, energy dispersion relation and filling factors will be covered. Relativistic quantum Hall effect is a newly discovered phenomena discovered in 2005 and is a very interesting subject. The way to approach it is similar to how integer quantum Hall effect was introduced. But before talking about Relativistic Hall effects, one may start to introduce graphene.

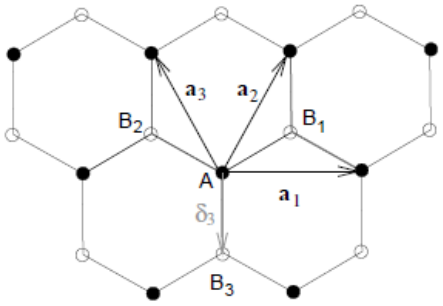

Figure 10: Honeycomb lattice with sublattices A and B [1]

.

5.1 Graphene

Graphene is one layer of carbon atoms in a honeycomb lattice (see figure 10) and it is a material with some extraordinary properties. It is both light weight and at the same time very strong, but not only that, it also conducts electricity just as good as copper [9][10]. These combined properties makes graphene an excellent choice for applications in a wide range of fields such as tissue engineering within medicine [11], components to improve Li-ion batteries [12] and composite materials to construct satellites, aircraft and cars [10]. Another interesting aspect of graphene is the dispersion relation for electrons and holes. The dispersion relation is linear at the Dirac points, which means that the effective mass for the electrons and holes is zero [10]. This will be described in this chapter in the picture of the tight binding model.

By using the tight binding model one can calculate the band structure [1, p. 105]. The tight binding model is based on the assumption that one have isolated atoms located at every lattice point. One also assumes that the amplitude of the electron wave functions to decay fast enough as one move away from the atom. Then one reduce the lattice constant such that the atomic wave functions from each neighboring atom starts to overlap each other. Then one can make an ansatz to the wave function inform of a Bloch function, which has to satisfy the Schrödinger equation.

In order to describe the honeycomb lattice, one has to divide it into two sublattices A and B. Then the wave function becomes a superposition of the wave function for each sublattice,

$$\psi_k(\vec{r}) = a_k\psi_k^{(A)}(\vec{r}) + b_k\psi_k^{(B)}(\vec{r}), \tag{144}$$

where $\psi_k^{(A)}(\vec{r})$ and $\psi_k^{(B)}(\vec{r})$ are Bloch functions,

$$\psi_k^{(j)}(\vec{r}) = \sum_{R_l} e^{i\vec{k}\cdot\vec{R}_l}\phi^{(j)}(\vec{r} + \vec{\delta}_j - \vec{R}_l) \tag{145}$$

and $\phi^{(j)}(\vec{r} + \delta_j - \vec{R}_l)$ are the atomic wave function centered at $\vec{R}_l - \delta_j$, where $\vec{R}_l$ is the vector pointing towards the elementary cell and $\vec{\delta}_j$ is the vector defining positions of B atoms inside the unit cell. Then one wants the find the solution to the Schrödinger's equation $H\psi_k = \epsilon_k\psi_k$, which can be done by multiplying it with ψ_k^*,

$$\psi_k^* H \psi_k = \epsilon_k \psi_k^* \psi_k. \tag{146}$$

Then plugging in equation (144) and multiplying out and writing it in matrix form for simplicity one get,

$$(a_k^*, b_k^*) H_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \epsilon_k (a_k^*, b_k^*) S_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{147}$$

where

$$H_k = \begin{pmatrix} \psi_k^{(A)*} H \psi_k^{(A)} & \psi^{(A)*} H \psi_k^{(B)} \\ \psi_k^{(B)*} H \psi_k^{(A)} & \psi_k^{(B)*} H \psi_k^{(B)} \end{pmatrix} \tag{148}$$

is the Hamiltonian matrix and

$$S_k = \begin{pmatrix} \psi_k^{(A)*} \psi_k^{(A)} & \psi^{(A)*} \psi_k^{(B)} \\ \psi_k^{(B)*} \psi_k^{(A)} & \psi_k^{(B)*} \psi_k^{(B)} \end{pmatrix} \tag{149}$$

is the overlap matrix.

One can calculate the energy bands by calculating the eigenvalues to the Schrödinger equation, found in the usual way by [1, p. 106]

$$det[H_k - \epsilon_k^\lambda S_k] = 0. \tag{150}$$

Now one ignores the overlap of $\psi^{(A)}$ and $\psi^{(B)}$ wave functions which results in the overlap matrix being equal to the unit matrix times N, the number of particles. What one pays attention to is the off-diagonal terms in the Hamiltonian matrix,

$$H_k^{AB} = \psi_k^{(A)*} H \psi_k^{(B)} = N t_k^{AB}, \tag{151}$$

where t_k^{AB} is the hopping term,

$$t_k^{AB} = \sum_{R_l} e^{i\vec{k}\cdot\vec{R}_l} \int d^2 r \phi^{(A)*}(\vec{r} - \vec{R}_k) H \psi^{(B)}(\vec{r} + \delta_{AB} - \vec{R_m}) \tag{152}$$

and δ_{AB} is a connecting vector, that connects a lattice site in sublattice A to a lattice point in sublattice B. The hopping term tells how the electrons interact with their neighbors [7, p. 222].

The goal is to descibe the band structure of graphene and to do that one only needs to consider hopping between the nearest neighbor. If one pick any lattice site on for example A, then it will have three neighbors B_1, B_2 and B_3 all with the same hopping amplitude given by

$$t = \int d^2 r \phi^{A*}(\vec{r}) H \phi^B(\vec{r} + \vec{\delta_3}). \tag{153}$$

B_3 do not correspond to any phase shift, since the vector $\vec{\delta_3}$ that corresponds to a shift is the same as the one to describe the location of B_3, hence they cancel. Then the other lattice sites have a shift that is

$$\vec{a}_2 = \frac{\sqrt{3}a}{2}(\vec{e}_x + \sqrt{3}\vec{e}_y) \tag{154}$$

$$\vec{a}_3 = \vec{a}_2 - \vec{a}_1 = \frac{\sqrt{3}a}{2}(-\vec{e}_x + \sqrt{3}\vec{e}_y). \tag{155}$$

Then the hopping term can be written as

$$t_k^{AB} = t\gamma_k^* = (t_k^{AB})^*, \tag{156}$$

where $\gamma_k^* = 1 + e^{ika_2} + e^{ika_3}$ and the exponentials are phase factors. Then one find the dispersion relation, which in turn gives the band structure,

$$\epsilon_\lambda(\vec{k}) = \lambda|t_k^{AB}| = \lambda t|\gamma_k|. \tag{157}$$

One interesting thing with graphene is that the valence band and the conduction band touch each other at [1, p. 107]

$$\pm\vec{K} = \pm\frac{4\pi}{3\sqrt{3}a}\vec{e}_x. \tag{158}$$

To find the relativistic Hamiltonian one can expand the phase factor, γ_k^*, around the points K and K',

$$\gamma_p^\pm \equiv 1 + e^{\pm i\vec{K}\cdot\vec{a}_2}e^{i\vec{p}\cdot\vec{a}_2} + e^{\pm i\vec{K}\cdot\vec{a}_3}e^{i\vec{p}\cdot\vec{a}_3} \tag{159}$$

$$\simeq 1 + e^{\pm 2\pi i/3}[1 + i\vec{p}\cdot\vec{a}_2] + e^{\mp 2\pi i/3}[1 + i\vec{p}\cdot\vec{a}_3] \tag{160}$$

$$= \gamma_p^{\pm(0)} + \gamma_p^{\pm(1)}. \tag{161}$$

The first term $\gamma_p^{\pm(0)}$ is equal to zero in the Dirac points K and K' and the expansion of $\gamma_p^{\pm(1)}$ to the first order term becomes,

$$\gamma_p^{\pm(1)} = i\frac{\sqrt{3}a}{2}\Big[(p_x + \sqrt{3}p_y)e^{\pm 2\pi i/3} + (-p_x + \sqrt{3}p_y)e^{\mp 2\pi i/3}\Big]$$

$$= \mp\frac{3a}{2}(p_x \pm ip_y). \tag{162}$$

This gives the Hamiltonian

$$H_p^\xi = \xi v(p_x\sigma_1 + \xi p_y\sigma_2), \tag{163}$$

where $v = 3ta/2\hbar$ and $\xi = \pm$ represents the different valleys K and K'. The relativistic Hamiltonian can then be written in a more compacted form [1, p. 108],

$$H_D^\xi = \xi v\vec{p}\cdot\vec{\sigma}. \tag{164}$$

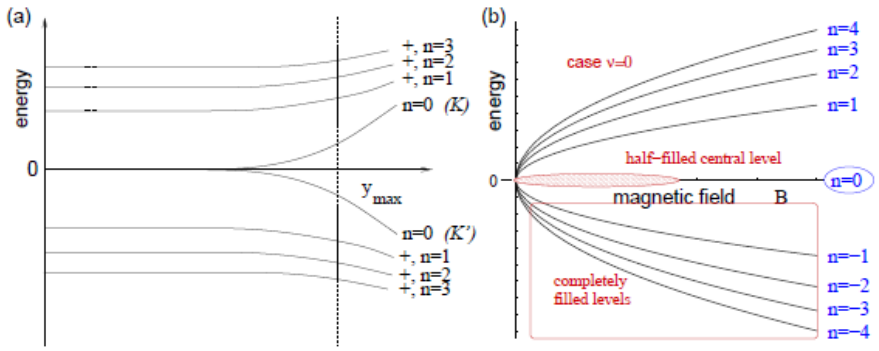

Figure 11: (a) shows the mass confinement for the relativistic Landau levels. As one moves closer to the edge the energy levels starts to diverge, for electrons is the energy bending upwards to y_{max}, while for holes its going downwards towards y_{min}. The evolution of the $n = 0$ level depends on the landscape. (b) shows how the hole states are completely filled while the $n = 0$ is partially filled and the electrons are unoccupied. [1]

5.2 Relativistic Quantum Hall effect

The relativistic quantum Hall effect in graphene can be analyzed in the same was as before in the Landau quantization picture, but here one need to take into account different charge carriers, electrons and holes, unlike before when only one charge carrier was considered. The problem here is the confinement potential in which one cannot use the same approach as one did for the usual integer Hall effect. If one would take the limit as the confinement potential goes to infinity, one would then confine the electrons, but the holes, which has opposite charge will not be confined or the other way around, if the confinement potential went to minus infinity [1, p. 61]. The solution to this is to use the mass confinement potential, in terms of Pauli matrices,

$$V_{conf}(y) = V(y)\sigma_3 = \begin{pmatrix} V(y) & 0 \\ 0 & -V(y) \end{pmatrix}. \tag{165}$$

The reason it is called mass confinement is that it acts as if it would be a mass for constant $V(y)$. The term $M\sigma_3^2$ is added to the relativistic Hamiltonian

$$H_D^m = v\vec{p}\cdot\vec{\sigma} + M\sigma_3 = \begin{pmatrix} M & v(p_x - ip_y) \\ v(p_x + ip_y) & -M \end{pmatrix} \tag{166}$$

this gives the energy spectrum

$$\epsilon_\lambda(\vec{p}) = \lambda\sqrt{v^2|p|^2 + M^2} \tag{167}$$

which is the dispersion relation for a relativistic particle with mass m. One can write the massive Dirac Hamiltonian in terms of ladder operators

$$H_D^m = \begin{pmatrix} M & v(\Pi_x - i\Pi_y) \\ v(\Pi_x + i\Pi_y) & -M \end{pmatrix} = \begin{pmatrix} M & \sqrt{2}\frac{hv}{l_B}a \\ \sqrt{2}\frac{hv}{l_B}a^\dagger & -M \end{pmatrix} \tag{168}$$

and find the eigenvalues which is shown in figure 11,

$$\epsilon_{\lambda n} = \lambda \sqrt{M^2 + 2\frac{\hbar^2 v^2}{l_B^2} n}. \tag{169}$$

Electrons and holes behaves like Dirac fermions, massless relativistic particles at the Dirac point in the Brillouin zone. This model is physical on length scales that is large in compared with the lattice constant. In this way electrons are confined to the sample [1, p. 62].

Another important property of relativistic quantum Hall effect is the filling factor, ν. The filling factor for relativistic Hall effect is

$$\nu = \pm 2(2n + 1). \tag{170}$$

This is different compared to the integer quantum Hall effect. Because in relativistic Hall effect the Landau levels are four-fold degenerate. There is a two-fold degeneracy of hills and valleys and another two-fold from spin degeneracy. This spin degeneracy should not be confused with the spin of the particle, but rather its due to the different lattices used to describe graphene. The extra term ± 2 comes from the fact that the energy levels are half filled at $n = 0$.

6 Conclusions

In this review many different theoretical subjects were mentioned. In order to get an understanding of quantum Hall effect, Landau quantization had to be introduced. The reason is that one has to explain the quantized cyclotron orbits that one observe during the experiment. No classical theory can explain the plateau pattern and why the longitudinal resistance disappear at these plateaus. One also had to analyze how particles behaves in very strong magnetic fields and treat them with quantum mechanics in order to get a good description. The quantized resistance also defines a new standard to resistance and the quantization is independent of the material used and only depend on the electron charge and Planck's constant, two fundamental constants of nature. Other applications to the quantum Hall effect is that one can use it to determine the fine structure constant, which is another fundamental constant. The field of quantum Hall effect is still a very active field of research, which provide deeper studies into this field. Classical Hall effect was discovered back in 1879 and integer quantum Hall effect was observed in 1980 and as late as 2005 the relativistic Hall effect was observed experimentally. The reason that the quantum Hall effects are discovered so late is probably because one need very sophisticated technology to achieve these conditions in a laboratory. Another recently observed quantum Hall effect which is not mentioned in this review is the Fractional Quantum Hall Effect (FQHE). Instead of the resistance taking values of integers, it takes values of fractions of e^2/h. There is still no theoretical description for experimentally observed results regarding the fractional quantum Hall effect.

7 Acknowledgments

I would like to thank my supervisor Anton Nedelin at the institution of theoretical physics at Uppsala university for making this project possible and providing me this very interesting and novel material to study.

References

[1] Mark O. Goerbig *Quantum Hall effect*, [http://arxiv.org/pdf/0909.1998v2.pdf], Laboratoire de Physique des Solides, CNRS UMR 8502 Universié Paris-Sud, France, 2009 [2015-05-26]

[2] Philip Hofmann *Solid states physics, An introduction*, WILEY-VCH, 1st Edition, 2008.

[3] Neil W. Ashcroft and N. David Mermin *Solid state physics*, BROOKS/COLE CENGAGE Learning, 1976.

[4] J.J Sakurai and Jim J. Napolitano *Mordern Quantum Mechanics*, PEARSON, 2nd Edition, 2011.

[5] David J. Griffiths *Introduction to electrodynamics*, PEARSON, 3rd Edition, 2008.

[6] David J. Griffiths *Introduction to Quantum Mechanics*, PEARSON, 2nd Edition, 2005.

[7] Michael P. Marder *Condensed matter physics*, WILEY, 2nd Edition, 2010.

[8] K. v. Klitzing and G. Ebert *Application of the Quantum Hall Effect in Metrology*, [http://iopscience.iop.org/0026-1394/21/1/004/pdf/0026-1394_21_1_004.pdf], Department of Physics. Technical University of Munich, D-8046 Garching, Federal Republic of Germany, 1984 [2015-06-08]

[9] The royal Swedish academy of sciences *Nobel prize in physics 2010*, [http://www.kva.se/sv/pressrum/Pressmeddelanden-2010/Nobelpriset-i-fysik-2010/], 2010 [2015-06-08]

[10] The royal Swedish academy of sciences *Scientific Background on the Nobel Prize in Physics 2010, Graphene*, [http://www.kva.se/globalassets/priser/nobel/2010/sciback_fy_10.pdf], 2010 [2015-06-08]

[11] Gaurav Lalwani, Allan M. Henslee, Behzad Farshid, Liangjun Lin, F. Kurtis Kasper, Yi-Xian Qin, Antonios G. Mikos and Balaji Sitharaman *Two-Dimensional Nanostructure-Reinforced Biodegradable Polymeric Nanoocomposites for Bone Tissue Engineering*, [http://www.ncbi.nlm.nih.gov/pmc/articles/PMC3601907/pdf/nihms450791.pdf], 2013 [2015-06-08]

[12] Fei Yao, Fethullah Gunes, Huy Quang Ta, Seung Mi Lee, Seung Jin Chae, Kyeu Yoon Sheem, Costel Sorin Cojocaru, Si Shen Xie and Young Hee Lee *Diffusion Mechanism of Lithium Ion through Basal Plane of Layered Graphene*, [http://pubs.acs.org/doi/pdf/10.1021/ja301586m], Department of Energy Science, BK21 Physics Division, Graphene Center, Sungkyunkwan Advanced Institute of Nanotechnology, Sungkyunkwan University, Suwon 440-746, Republic of South Korea, 2012 [2015-06-08]