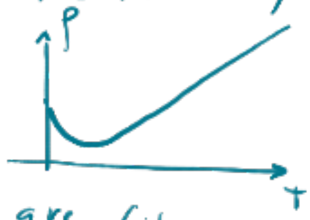
drude theory

The origin of Electron Conduction in normal metals.

How can we quantify when a metal is a good conductor of heat vs. electricity. Enter the Wiedemann-Franz law: $K/\sigma \cong \text{const} \simeq 2\text{-}3 \cdot 10^{-8} \frac{W\Omega}{K^2}$

Matthiessen also noted that at low T the resistivity

$$\approx \quad \frac{1}{\sigma(T)} = \frac{1}{\sigma_{imp}} + \frac{1}{\sigma_0(T)}$$



- Drude assumed that some electrons are "free" and detach itself from ions. (it was around 1900!)
- The electrons can be treated as dilute gas without Coulomb int. among them.

- Recall from the gas theory:

$$\langle \dot{v} \rangle = -\frac{e}{m_e} E - \frac{1}{\tau} \langle v \rangle$$

local electric field ≠ with ext. $\bar{E}$

- in the steady regime

$$\langle \dot{v} \rangle = 0 \quad \Rightarrow \bar{v} = \text{the drift velocity} = -\frac{e}{m} \tau \bar{E}$$

From this the current $\bar{j} = -ne\langle v \rangle = -n\frac{e^2}{m}\tau\bar{E}$

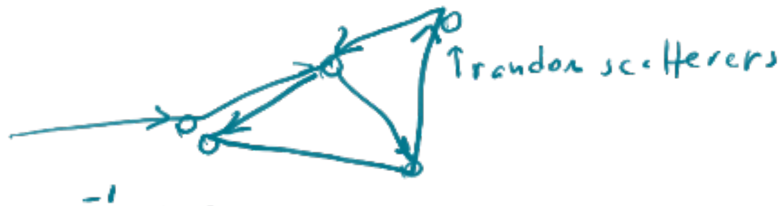
$$\bar{j} = \sigma\bar{E} \quad \text{where}$$

$$\sigma = \frac{ne^2}{m}\tau$$

Do problem 7.1

The question is what is τ: the mean free path

$$\tau = \frac{\ell}{\bar{v}}$$



$$\ell^{-1} = \nu\sigma$$

density of scatterers

cross-section

We also avoid small angle scattering events and only pay attention to the large ange ones. $\sim 90^\circ$

$$\ell_{TR}^{-1} = \nu \int d\Omega\,(1-\cos\theta)\frac{d\sigma}{d\Omega} \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

Note, if the scattering is isotropic $\ell = \ell_{TR}$ (normal!) but if we have forward scattering $\ell_{TR} \gg \ell$

When e^- recocheting between atoms:

$$\ell_{TR} \sim \ell \sim a \text{ , and } \frac{1}{2} m_e \bar{v}^2 = \frac{3}{2} k_B T$$

(here Drude knew nothing about fermions!)

for RT = 23 meV = 300 K $\bar{v} \sim 10^7$ cm/s

Now we have $\sigma = \frac{ne^2\tau}{m_e} = \frac{ne^2}{m}\frac{\ell}{\bar{v}} = \frac{ne^2 a}{10^7 \text{ cm/sec}}$ $a \sim 10^{-7}$ cm

$$\Rightarrow \rho \equiv \frac{1}{\sigma} \sim 10^{-6}\left(\frac{T}{T_{RT}}\right)^{1/2} \Omega\cdot\text{cm}$$

For thermal transport:

$$\vec{J}_Q = -\kappa_e \nabla T$$ where kinetic theory predicts:

$$\kappa_e = \frac{2}{3}\tau \bar{v}^2 C_e$$

$\uparrow$ electronic specif heat $= \frac{3}{2} n k_B$

From $\cancel{\frac{1}{2}} m_e v^2 = \cancel{\frac{3}{2}} k_B T \Rightarrow v^2 = \frac{3 k_B T}{m_e}$

$$\kappa_e = \cancel{\frac{2}{3}} \tau \cancel{\frac{3}{2}} n k_B \cdot \frac{3 k_B T}{m_e} = \frac{3 n k_B^2 T}{m_e}\tau$$

Finally, we can calculate the Lorentz number:

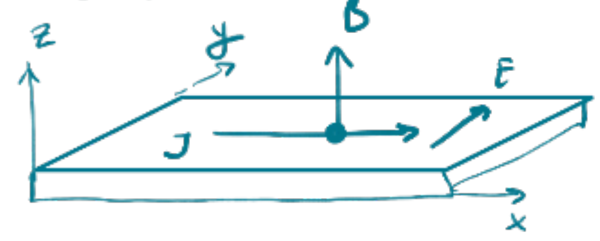
$$L \equiv \frac{\kappa_e}{\sigma T} = \frac{3 \cancel{n} k_B^2 \cancel{T}}{\cancel{m}} \frac{\cancel{\tau}\ \cancel{m}}{\cancel{T} \cdot \cancel{n} e^2 \cancel{\tau}} =$$

$$= 3\left(\frac{k_B}{e}\right)^2 \sim 2.2 \cdot 10^{-8} \frac{V^2}{K^2}$$

Again this is correct!

But now we move to the difficulties:

Hall effect:



$F_L = \frac{q}{c} \bar{v} \times \bar{B}$

$q = -e$

$\langle \dot{\bar{v}} \rangle = -e\ \bar{E} - e\ \langle \bar{v} \rangle \times B - \langle \bar{v} \rangle$

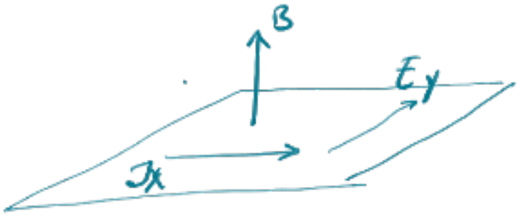
$$\langle v\rangle = -\frac{\quad}{m} \quad - \frac{\quad}{mc}\langle v\rangle \times B \qquad \tau$$

$\bar{B} = B\hat{z}$, in steady state $\langle \dot{v}\rangle = 0 \Rightarrow$

$$\boxed{\begin{gathered} E_\mu = \rho_{\mu\nu} j_\nu \\ \rho_{\mu\nu} = \begin{pmatrix} \sigma^{-1} & +B/nec & 0 \\ -\frac{B}{nec} & \sigma^{-1} & 0 \\ 0 & 0 & \sigma^{-1} \end{pmatrix} \end{gathered}}$$

inverse Drude cond.

$$\rho_{xy} = -\rho_{yx} = -\frac{B}{nec} \leftarrow$$ HALL RESISTIVITY



$$J_x = \frac{1}{\rho_{xy}} E_y \Rightarrow E_y = \rho_{xy} j_x$$

and it will balance the lorentz force

In this classical case:

$$r_H = \frac{\rho_{xy}}{B_z} = \frac{-\cancel{B_z}}{nec}\cdot\frac{1}{\cancel{B_z}} = \frac{1}{nec}$$

HALL COEFFICIENT

electron density

(in SI units $r_H = -\frac{1}{ne}$)

Most important for experiment! $\Rightarrow$ density of carriers and SIGN OF CARRIES

So the order of magnitude is correct but sign can be wrong, e.g. in MR.

Surprisingly, but diagonal terms $\neq B$!?
so we say there is no magnetoresistance in the classical case, as there is no particle interference between pathes for the final state, e.g. Aharonov - Bohm effect

But $\sigma_{\mu\nu} \equiv \frac{1}{\rho_{\mu\nu}}$ does have diagonal terms which depend on B!

Exers. 7.2.

Once we include quantum statistics we get a new energy scale $E_F \sim 1\text{-}3\,eV \approx 10^4\,K$
it is kind of Debye T_Θ for phonons:
quantum effects freeze out e^- contribution to the heat transport.

Another error: $l_{m.f.p}$ is much longer than lattice parameters.

$$l_{mfp} \gg a. \Rightarrow \tau = \frac{l}{\langle v \rangle}$$

← is really large

but for fermions it's large too

So accidently τ is correct!

Solve problem: 7.3.

Independent electron model

Boltzmann equation and relaxation time

Boltzmann came up with a very general expression for all kinds of transport. This approach is standard in modern condensed matter physics since it describes the changes in the distribution function upon application of any perturbation.

* Relaxation time approximation:
Let's consider a single band and lets call $f(\bar{k}, t)$ the probability that a state with $\bar{k}$ occurs at time t. Normally it would be the Fermi-Dirac distribution in equilibrium, since $E(\underline{k})$

If the solid is not at equil. we have different f

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{field} + \left[\frac{\partial f}{\partial t}\right]_{scatt.}$$

$$\left[\frac{\partial f}{\partial t}\right]_{field} = \lim_{\Delta t \to 0} \frac{f(k, t+\Delta t) - f(k,t)}{\Delta t} \Bigg/_{field}$$

Note k remains unchanged. Change in field during the time Δt.

e.g. in time Δt the state $k \to k + \frac{eE}{\hbar} \cdot \Delta t \longrightarrow$

$$f(k, t+\Delta t) = f(k + eE\Delta t/\hbar, t) = f(k,t) + eE \cdot \nabla_k f \, \Delta t/\hbar \Rightarrow$$

$$\boxed{\left[\frac{\partial f}{\partial t}\right]_{field} = + \frac{eE \, \nabla_k f}{\hbar}}$$

As for the scattering rate we take very generally

$$\left[\frac{\partial f}{\partial t}\right]_{scatt} = -\frac{f(k,t) - f_0(k)}{\tau}$$

$f_0 \Rightarrow$ Fermi-Dirac distribution

$\tau \leftarrow$ relaxation time

What we want is to understand how the tendency to equilibrium gets restorted after we applied the field, e.g. $\bar{E}$. It's simple

If $f > f_0 \Rightarrow \left(\frac{\partial f}{\partial t}\right)_{scatt} > 0$ and $f \uparrow$.

Now suppose $f(k,0)$ $t=0$ is in non-equilibrium if $E=0$ we get

$$\frac{\partial f}{\partial t} = -(f - f_0)/\tau \quad \text{(see prev. page)}$$

$$\hookrightarrow f(k,t) = f_0(k) + [f(k,0) - f_0(k)]e^{-t/\tau}$$

the distribution relaxes exponentially fast in time and for small τ it's almost in equilibrium.

Now we turn on the field:

$$\boxed{\frac{\partial f}{\partial t} = \frac{e}{\hbar}\bar{E}\cdot\nabla_k f - \frac{f - f_0}{\tau}}$$

The famous Boltzmann transport equation

Take $f(\bar{k},t) = f_0(\bar{k}) + f_1(\bar{k},t)$

$f_1 \uparrow$ small deviation from equilibrium

$$\frac{\partial\left(f_0(k) + f_1(k,t)\right)}{\partial t} = \frac{e}{\hbar} E \cdot \nabla_k \left(f_0 + f_1\right) - \frac{\left(f_0 + f_1\right) - f_0}{\tau}$$

$$\boxed{\frac{\partial f_1}{\partial t} = \frac{e}{\hbar} E \cdot \underbrace{\nabla_k f_0} - \frac{f_1}{\tau}} =$$

$$= e\bar{E} \cdot v \frac{\partial f_0}{\partial E} - \frac{f_1}{\tau}$$

$$\nabla_k f_0 = \frac{\partial f_0}{\partial E} \underbrace{\nabla_k E}_{= \hbar \bar{v}}$$

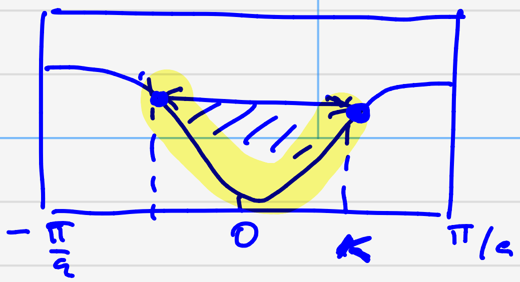
$t = 0$: we turn on the field and the solution is

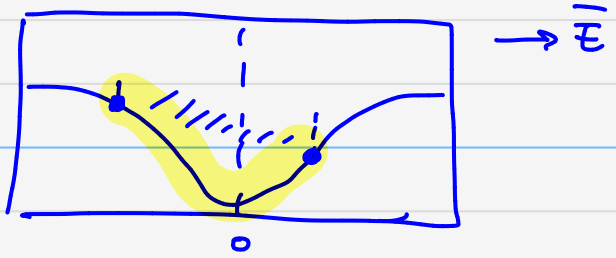
$$f_1(k,t) = \underbrace{e E \cdot \bar{v} \tau \frac{\partial f_0}{\partial E}} \left[1 - e^{-t/\tau}\right]$$

for $t \gg \tau$ the electron system reaches the steady state:

$$f(k) = f_0(k) + f_1(k) = f_0(k) + e\bar{E} \cdot \bar{v} \tau \frac{\partial f_0}{\partial E}$$

Since $\frac{\partial f_0}{\partial E} < 0$





Lets now study what average momentum does

$$\hbar\bar{g} = \sum_{\hbar} \bar{g}\, f(\bar{g},t)/N$$

reminder

$$\boxed{\bar{g} = \text{average Momentum}}$$

going back to

$$\underset{\substack{\text{over all}\\ \text{States in}\\ \text{the band}}}{\sum} \hbar\bar{g} \times \Bigg| \quad \frac{\partial f}{\partial t} = \left[\frac{\partial f}{\partial t}\right]_{field} + \left[\frac{\partial f}{\partial t}\right]_{scatt.}$$

$$\frac{d\left(\hbar\bar{g}\right)}{dt} =$$

$$= \left(\frac{d\,\hbar\bar{g}}{dt}\right)_{field} \quad ?$$

for each electron

$$\frac{d\,\overset{\text{Momentum}}{\boxed{\hbar\bar{g}}}}{dt} = -eE \quad \Rightarrow \sum_{\substack{\text{over}\\ \text{states}}} \frac{d\,\hbar g}{dt} = \underbrace{-eE}$$

and if τ is independent of $\bar{k}$

$$\left(\frac{d\,\hbar\bar{g}}{dt}\right)_{scatt} = -\frac{\hbar\bar{g}}{\tau} \quad \text{as the result}$$

we have

$$\frac{d\,\hbar\bar{g}}{dt} = -eE - \frac{\hbar g}{\tau}$$

and in the steady stat $t \gg \tau$

$$\hbar\bar{g} = -eE\tau \neq 0 \,!$$

Unlike the case when electrons are at equilibrium

This means for metals only band crossing E_F contribute to mobility μ.

Consider a hypothetical cubic xtal:

$\mu_{ij} = \mu$ scalar

for $i = j$ $v_i v_j$ and $-v_i v_j$ occur equally so all those terms will cancel out

Also v_x^2, v_y^2, v_z^2 are the same ⇒

$$\mu_{xx} = \mu_{yy} = \mu_{zz} = \mu \Rightarrow$$

$\bar{v} = -\mu E$ where $\mu = \frac{1}{3}(\mu_{xx} + \mu_{yy} + \mu_{zz})$

$$\boxed{\mu^{e^-} = -\frac{e}{3N} \sum \tau v^2 \frac{\partial f_0^{electrons}}{\partial E}}$$

Since $\frac{\partial f_0}{\partial E} < 0 \Rightarrow \mu > 0$ for electrons, but $\bar{v} \uparrow\downarrow \bar{E}$

For non-cubic xtals it's also ok but

$\tau = \tau(\bar{k})$

Similarly for hole we can write down

SHOW THIS!

$$\boxed{\mu^{holes} = \frac{e}{3N} \sum \tau v^2 \frac{\partial f_0^{holes}}{\partial E}}$$

$\frac{\partial f_0^{holes}}{\partial E} > 0$ and $\mu > 0$ but $\bar{v} \parallel \bar{E}$

Physical meaning of $\bar{g}$:

suppose at $t=0$ $g=0$ then for the time τ it accelerates and becomes $-eE\tau$ but returns back to $g=0$ after scattering

That is why τ is mean free time.

Mobility: In the steady state $\bar{v} \sim E$ ← average velocity

and thus:

mobility

$$\bar{v}_i = -\sum_j \mu_{ij} E_j$$

often called electron drift velocity

Lets try to obtain the expression for μ_{ij}

1. Start with

$$\bar{v}_i = \frac{1}{N} \sum_{\text{states}} v_i(k) f(k)$$

\# of electrons in the band

and $f(k) = f_0(k) + eE\cdot\bar{v}\tau\frac{\partial f_0}{\partial E}$

$$= \underbrace{\frac{1}{N}\sum v_i(k) f_0(k)}_{=0} + \frac{1}{N}\sum v_i\, eE\bar{v}\tau\frac{\partial f_0}{\partial E}$$

$$= \frac{e}{N}\sum_{\text{state}} v_i \bar{E}\cdot\bar{v}\,\tau\frac{\partial f_0}{\partial E} \Rightarrow$$

$$\mu_{ij} = -\frac{e}{N}\sum_{\text{states}} \tau v_i v_j \frac{\partial f_0}{\partial E}$$

since $\frac{\partial f_0}{\partial E}$ spikes only at E_F only the state at the chem. potential matter.

Electrical Conduction in bands

$\bar{j} = q n \bar{v}$ in E&M

In solid state physics:

the carrier density $n = \frac{f(k)}{V}$

$$\bar{j} = -\frac{e}{V} \sum_{\text{states}} f(\bar{k}) v(\bar{k})$$

Sum over states can be done like 1-band at the time and $\sum f(\bar{k}) \bar{v}(\bar{k})$ over states in the band can be identified as $N \cdot \bar{v}$

N = # of e^- in the band, $\bar{v}$ - average velocity

So: $$j = -e \sum_{\text{bands}} \frac{N}{V} \cdot \bar{v} = -e \sum_{\text{bands}} n \bar{v}$$

where n is the contribution of the band to total electron concentration.

Recall for the steady state $\bar{v}_i = -\sum_j \mu_{ij} E_j$

$$\boxed{j_i = e \sum_{\text{bands}} \left[n \sum_j \mu_{ij} E_j \right]}$$ and

remembering Ohm's law $J = \sigma E \Rightarrow$

$$j_i = \sum_j \sigma_{ij} E_j$$ where

$$\boxed{\sigma_{ij} = e \sum_{\text{bands}} n \mu_{ij}}$$ the conductivity tensor

Dynamics of electrons in external field

- The motion of electrons is defined by the shape of it's Fermi surface.
- i.e. for each point on the $E_i(\bar{p}) = E_{i\,Fermi}$ we need to know the dispersion curve for a given direction.
- Almost always we can consider the motion in a quasiclassical limit b/c $\lambda_B \sim 2\pi\hbar/p_F$ with $p_F \sim \frac{2\pi\hbar}{a}$ we get $\lambda_B \sim a$, while the mean free path for e^- is $\gg a$.
- Group velocity of e^- is always $\perp$ to the $E_i(p)$: $\frac{\partial E}{\partial p} = \bar{v} = \nabla_p E$ and in general is non-collinear to its momentum.

Lets study the case when we apply electric field $\bar{E}$ when $T = 0K$ (no scattering off phonons). Lets assume the direction of $\bar{E} \parallel [001]$

Lets assume we have only one e^- at the zone center: Γ point, where $p=0$ and $K=0$

now I turn on the field E
and my e^- experiences $\bar{F} = -e\bar{E}$
and $p = |e| E \cdot t$

Mass of electron in solids

In classical mechanics "the real" mass is defined as

$$m_0 = F/a$$

In a crystall we have forces from ions which are not known in general.

But we can define the characteristic parameter called effective mass such as

$$m_0 \cancel{\frac{dv}{dt} = \sum (F_{int}} + F_{ext}) \Rightarrow m^* \frac{dv}{dt} = F_{ext}.$$

Lets compare it to the arb. dispersion eqn:

$$\frac{dv_i}{dt} = \frac{dv_i}{dp_k} \frac{dp_k}{dt}$$

So the i^{th} component of the eqn. is

$v_i = \frac{dE}{dp_i}$ and taking into account

$\frac{dp_k}{dt} = F_k$ we get $\frac{dv_i}{dt} = \frac{\partial^2 E}{\partial p_k \partial p_i} F_k$

$$\boxed{\frac{\partial^2 E}{\partial p_k \partial p_i} = \frac{1}{m^*_{ij}}}$$

m^*_{ij} is called the effective mass tensor.

Few notes:

1) Diagonal terms m^*_{ii} on the isoenergy surface are positive e.g. for a sphere $E = p^2/2m^* \rightarrow m^* = \left(\partial^2 E/\partial p^2\right)^{-1} > 0$ b/c $\frac{\partial E}{\partial p} > 0$ for electrons $\left(\frac{\partial E}{\partial p} = v < 0 \text{ for holes}\right)$

2) $\bar{a}$ the acceleration is not collinear with F

3) Different k_s for different positions on the Fermi surface corresponds different m^* except for the extremal points.

4) The process of relaxation depends on how inelastic is the scattering

e.g. for many scattering events within τ the motion of electrons or its phase has many resets. And as such one can think of it as a motion in the viscous enviroment.

The magnitude of the relaxation time defines the boundary when one can talk about quasiparticels: e.g. from $\Delta E \cdot \tau \approx \hbar$ is possible to talk about the quasiparticles if $\Delta E \ll E_F$

if we have a very disordered enviroment or strongly correlated "Bad" metal where $\ell = v_F \cdot \tau \sim a$ (the so called Ioffer-Rögel limit)

$$\Delta E \sim \frac{\hbar}{\tau} = \frac{\hbar v_F}{\ell} = \frac{10^{-34}\, 10^{6}\, J}{10^{-10}} = 10^{-18} J \sim 6 eV$$

WITH THIS INFO LETS LEARN ABOUT
ELECTRON – PHONON SCATTERING

1. Quasimomentum of e^- doesn't change when it moves in an ideal periodic lattice

2. It can change only interactions with NON periodic part of the potential.

e.g. phonons, defects, dislocations, domain boundaries, surface, interface, grain boundaries

AND e – e interaction or e-e scattering

3. we have inelastic and elastic processes.

Now to the phonons: phonon is an elastic wave of deformation as the result we have an additional periodicity to the potential on which electrons can scatter off.

The effect is connected to the # phonons the electron can scatter.

Note: Similar to the phonon story during the scattering act we should consider both N-process and U-process

↑ no crystal momentum

↑ with crystal momentum.

But unlike phonons which has no chemical potential and thus no Fermi surface.
Electrons scattering is strongly bound to the topology of Fermi surface.

Electrical conductivity

Basics: in absence of ∇T and diffusion

$$\bar{j} = \underset{\uparrow\ \text{electrical conductivity}}{\sigma} \bar{E}$$

In isotropic cas $\bar{j} \parallel \bar{E}$ and σ is a scalar
But we know that almost all real crystals have anisotropy of some sort, in this case:

$$j_i = \sum_k \underset{\uparrow\ \text{tensor!}}{\sigma_{ik}} E_k \quad \Rightarrow \quad E_i = \sum_k \underset{\uparrow\ \text{resistance tensor}}{\rho_{ik}} j_k$$

the connection $\sigma_{ik} \rho_{ik} = \delta_{ik}$
also $\rho_{ik} = \rho_{ki}$

The components can be determined by measuring electric field along crystal axis 1, 2, 3

along 1: e.g $E_1 = \rho_{11} j_1$ $E_2 = \rho_{21} j_1$ etc

along 2: $E_1 = \rho_{12} j_2$ $E_2 = \rho_{22} j_2$ etc

In cubic xtal all non diagonal terms $= 0$
For a single axis materials:

$$\rho_{11} = \rho_{22} \neq \rho_{33}$$

Electrical conductivity in the model of Fermi carriers

- Lets assume we deal only with close FS
- and the only scatterers are <u>phonons</u>.
- Also lets assume the FS is close to spherical e.g. K, Li Na...

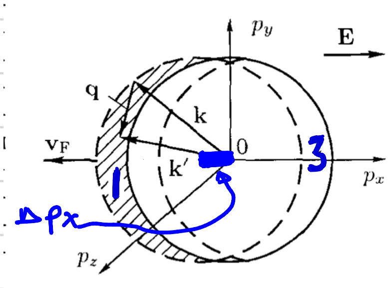
$E(p) = \frac{p^2}{2m^*}$, τ – time between scattering

$W_{k,k'}$ is the probability of scattering from $k \to k'$

: $W_{k,k'} \sim \frac{1}{\tau}$. After that the electron moves $\ell = v_F \cdot \tau$.

Apply E along x.

All electrons get extra $\Delta p_x = -eE_x\tau$



<u>The whole distribution</u> moves in the opposite direction by Δpx.

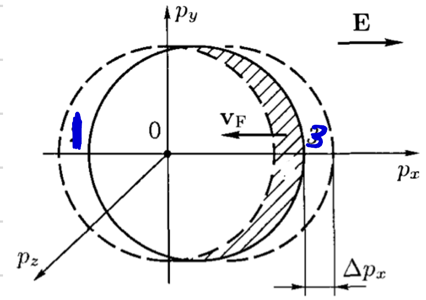
Without $\bar{E}$ we have no j since for each e^- with $+\bar{p}$ we have e^- with $-\bar{p}$

With application of E central symmetry is broken.

Now we have uncompensated charge in ① and ③

On the FS for electron we have $m^* > 0$ and $v_F \perp$ to the FS. Electrons in 1 and 3 create a flow of electrons with $v \sim v_F$ along $\bar{F} = -e\bar{E}$ + The higher $\bar{E}$ the large the volume.

Now consider holes: For hole $\frac{dp}{dt} = -e\bar{E}$



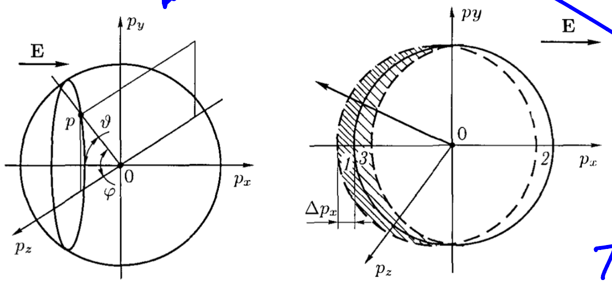
In volume ① we have e^-

② we deplete electrons

As the result, e^- with $v \sim v_F$ in ① and ③ create current in the same direction as electrons

As the result in metals with electrons and holes, current is a sum of e^- and h.

Electrical Conductivity



To derive the equation for σ, consider the spherical coordinates. Since $\Delta p \ll p_F$ any particle in 1 and 3 is along p and $v \sim v_F$. The projection of v onto $\vec{E}$ is $p\cos\theta / m^*$, since volumes ① and ③ are practically identical, we calculate the current in ①

- inside $dV_p = p^2 \sin\theta \, dp \, d\theta \, d\varphi$ we have

$$\Delta n = \frac{\underbrace{2}_{\text{spin}\uparrow\text{ and }\downarrow} dV_p}{(2\pi\hbar)^3} \text{ electons}$$

The contribution of those electrons into current

$$dj = e\Delta n v = e \frac{2p^2 \sin\theta \, dp \, d\theta \, d\varphi}{(2\pi\hbar)^3} v_F \cos\theta$$

The total current created by e^- in the volumes 1 and 3 thus:

$$j = 2 \cdot \frac{2ev_F}{(2\pi\hbar)^3} \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi \int_{p_F}^{p_F + \Delta p\cos\theta} p^2 \sin\theta \cos\theta \, dp =$$

$$= \boxed{j = \frac{8\pi e p_F v_F \Delta p_x}{3(2\pi\hbar)^3}}$$

Show this

$4\pi p_F^2$ = the area of the isoenergy surface for e^- and since $\Delta p_x = eE\tau = e.lmE/p_F$

$$\boxed{j = \frac{2}{3}\,\frac{e^2 S_F \ell}{(2\pi\hbar)^3}\,E}$$ and from $j = \sigma E \Rightarrow$

$$\boxed{\sigma = \frac{2}{3}\,\frac{e^2 S_F \ell}{(2\pi\hbar)^3}}$$ this eqn. is known as the Lifshits eqn.

It works for any dispersion as long as $E(p) = f(p^2)$ and f has no singularities.

Lets consider the simplest case $E = p^2/2m^*$ (BTW the linear dispersion is also can be of this kind, i.g. $E^{\pm} = \pm\alpha(p^2)^{1/2}$!) for whic $p=0$ is a singularity.

Lets rewrite eqn for σ as:

$$S_F = \frac{3\Delta_F}{p_F} \Leftarrow \left(\text{indeed } S_F = 4\pi p_F^2 \text{ and } \Delta_F = \frac{4}{3}\pi p_F^3\right)$$

the volume in momentum space

Also recal for isotropic dispersion $p_F = m^* v_F$

We get $$\sigma = \frac{2e^2\Delta_F\ell}{(2\pi\hbar)^3 m^* v_F} = \frac{e\tau}{m^*}\,\frac{2\Delta_F}{(2\pi\hbar)^3}$$

↑ this is concentration of e^- thus we can write down

$$\boxed{\sigma = \frac{e^2 n\tau}{m^*}}$$

Which is exactly the famous formula $\sigma = en\mu$ known as Drude formula.

clearly $\frac{e^{\cancel{2}}\cancel{n}\tau}{m^*} = \cancel{e}\cancel{n}\mu \Rightarrow \boxed{\mu = \frac{e\tau}{m^*}}$ Mobility

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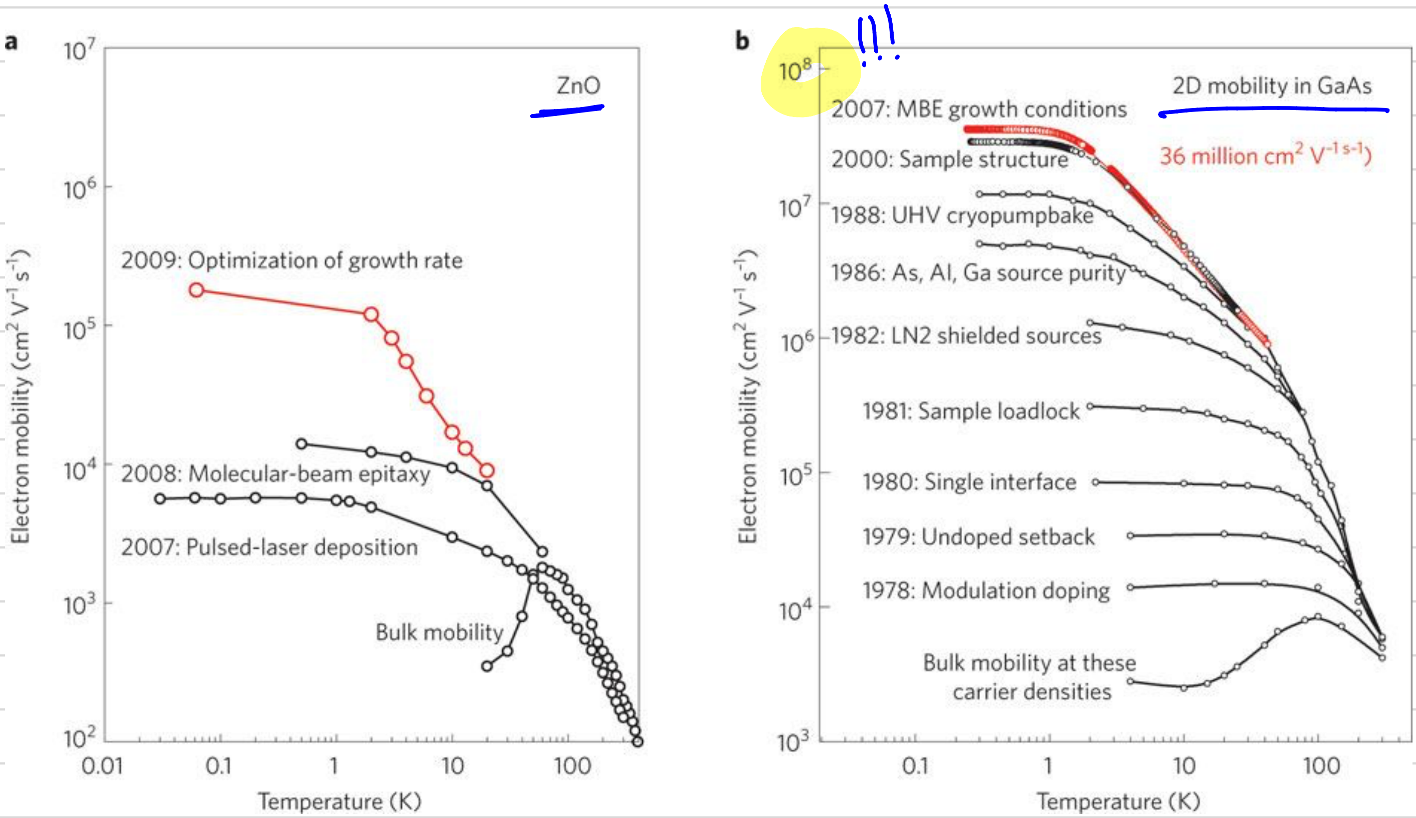
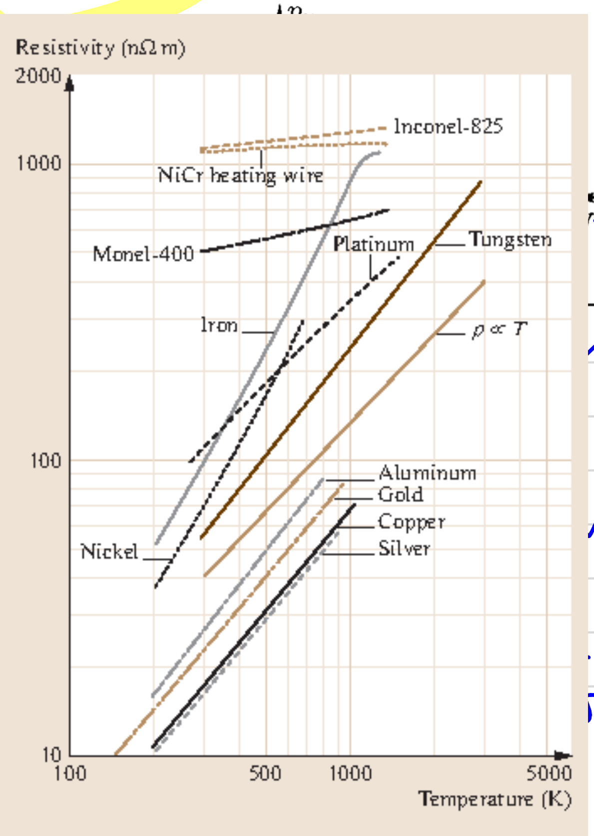


~~Electrical conductivity in the electron-hole model~~

Ignore This! →



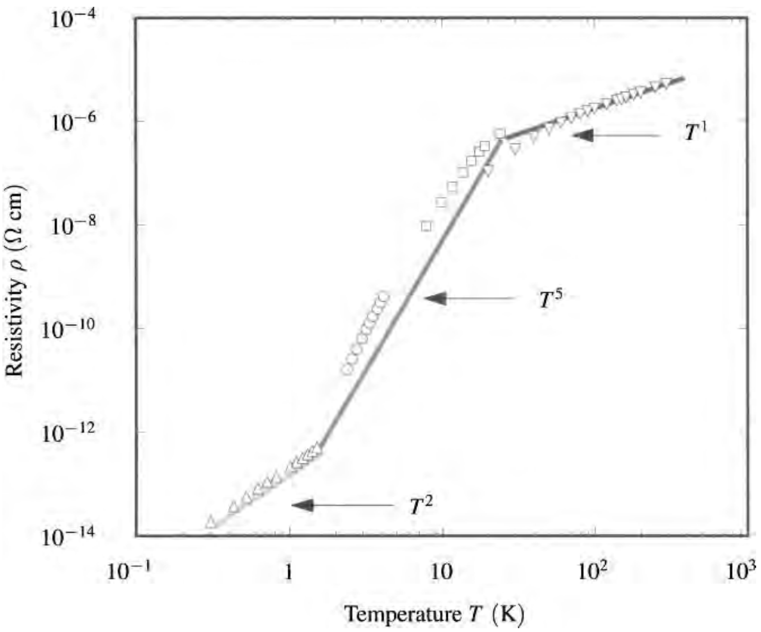


Figure 18.1. Resistivity of potassium from 0.1 to 300 K. At the lowest temperatures, the resistance varies as T^2, from 2 to 20 K, it varies as T^5, and from 100 K upwards, it varies as T^1. [Source: Bass et al. (1990).]

TEMPERATURE DEPENDENCE OF CONDUCTIVITY

Clearly the T-dep. comes from the T-dependent $\tau(T)$. In this theory we consider that the only important processes are those when $\Delta p \sim p$ or the electrons is transfered from ① in ② and as such we need the U-process.

The time which takes for the e^- to move from ① → ② is called the transport relaxation time.

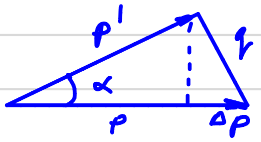
When $T > T_D$ e^- and phonons have momentum which $\sim \frac{\pi}{a} \cdot \hbar$. The # of scattering phonons for $T > T_D$:

$T > T_D$ | $\sim T$, thus $\tau_t \sim \frac{1}{T}$ and $\sigma \sim \tau_t \sim \frac{1}{T}$
and $\rho \sim T$

But when the momentum of phonons is small $T \ll T_D$ we have a problem to move from ① → ②. which now can only happen as the result of diffusive motion. (see fig. on p. 11)
$k \to k'$

As the result for each scattering act e^- with $\vec{p}$ with phonon with $\vec{q}$ the projection of of e^- momentum on to the direction of initial motion is:

$$\Delta p = p(1 - \cos\alpha)$$



$\Delta p = (1 - \cos\alpha) p$
$\cos\alpha \sim 1 - \alpha^2/2$
$\alpha = q/p$

So we get $\Delta p = p(1 - \cos\alpha) = \frac{\alpha^2 p}{2} \simeq \frac{q^2}{2p}$

Recal that $p \sim \pi \frac{\hbar}{a}$, and the momentum of phonons is given by $\hbar\omega_{ph} = v_a q \ll k_B T \Rightarrow q \cong k_B T / v_a$

and also recal that $T_D \sim \frac{\hbar \omega^{max}}{k_B}$

for ω^{max} we have $\lambda_{min} \simeq 2a \Rightarrow$

$$T_D \approx \frac{\hbar \omega^{max}}{k_B} = \frac{2\pi \hbar v_a}{2a\, k_B} \propto \frac{v_a p}{k_B}$$

and as the result we have

$$\Delta p \approx \frac{q^2}{2p} = \frac{k_B^2 T^2 v_a^2 p}{2 v_a^2 k_B^2 T_D^2} \simeq \left(\frac{T}{T_D}\right)^2 p$$

To scatter e^- we must turn in into the angle $\pi/2$ and the projection of the $\Delta p \sim p_F \Rightarrow$

The number of acts like this scales with $\frac{p}{\Delta p} \sim \left(\frac{T}{T_D}\right)^2 \sim \frac{1}{T^2}$

The number of available phonons for $T \ll T_D$ $\sim T^3$

$T \ll T_D$: Thus for the scattering of e^- on phonons with $\omega \ll \omega^{max}$

$$\tau_t \sim \left(\frac{1}{T^2}\right) \cdot \left(\frac{1}{T^3}\right) \Rightarrow \boxed{\begin{array}{l} \sigma \sim \frac{1}{T^5} \\ \text{and } \rho \sim T^5 \end{array}}$$

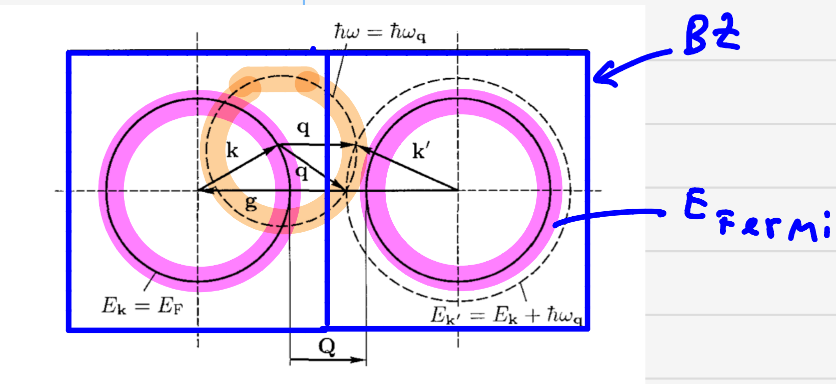
For the case of spherical FS and very specific spherical distribution of phonon momentum there is a general formula called the

Bloch - Grüneisen formula.

$\equiv \theta_D$

$$\rho_{\text{el-ph}} = \alpha_{\text{el-ph}} \left(\frac{T}{\theta}\right)^n \int_0^{\theta/T} \frac{x^n}{\left(e^x - 1\right)\left(1 - e^{-x}\right)} dx,$$

U - process in ē-phonon scattering



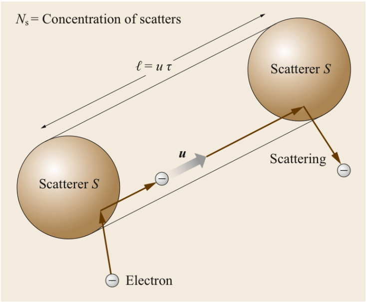


Fig. 2.1 Scattering of an electron from a scattering center. The electron travels a mean distance $\ell = u\tau$ between collisions

Here we have: $E(k') = E(k) + \hbar\omega_q$

and $\hbar k' = \hbar k + \hbar q + \hbar g$ ← xtal momentum

What is important, for a given k there will be a specific phonon with q, such that

$$\omega(q) = E(k') = E(k) + \hbar\omega$$

↑ isoenergy surface

But unlike N-process the intersection will dissappear at some $|q| = Q$.

The closer E_F's in the neighboring BZ are the more important is U-process.

e.g. in alkali metals U-process is dominant event at T = 20K!

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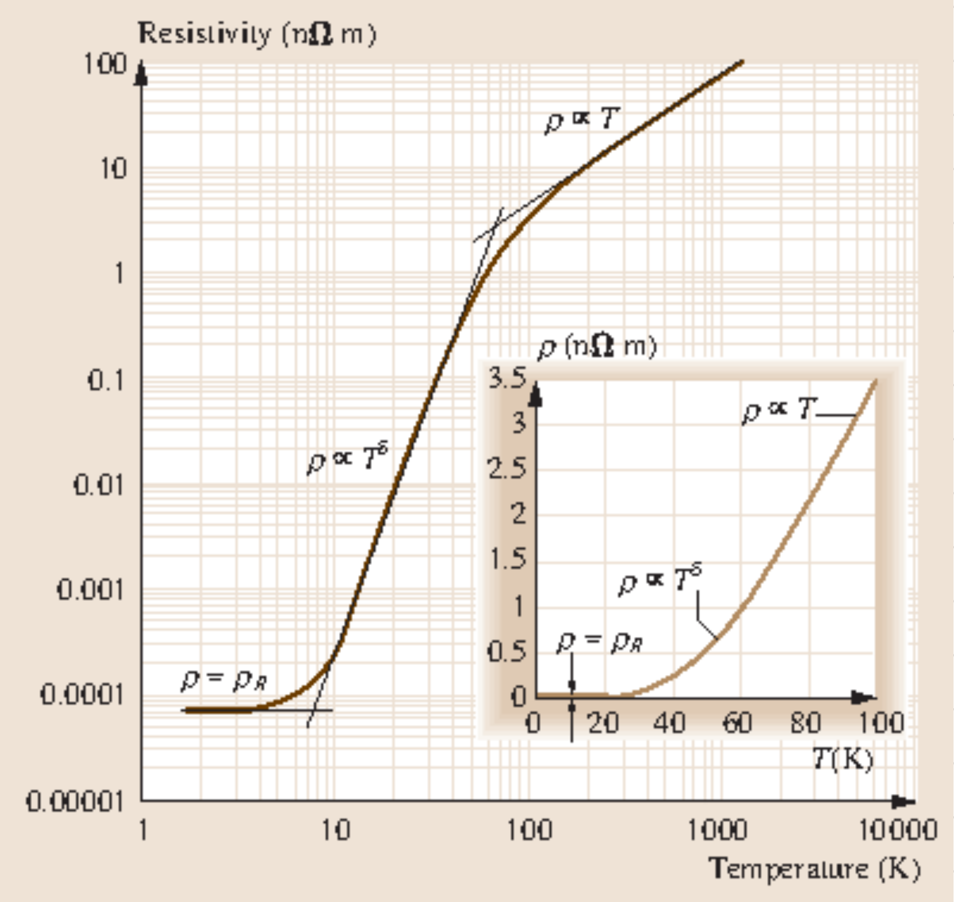


The resistivity of copper from low to high temperatures (near its melting temperature, 1358K) on a log–log plot. Above about 100K, ρ∝T, whereas at low temperatures, ρ∝T5, and at the lowest temperatures ρ approaches the residual resistivity ρR. The inset shows the ρ versus T behavior below 100K on a linear plot. (ρR is too small to see on this scale.)

Matthiessen rules

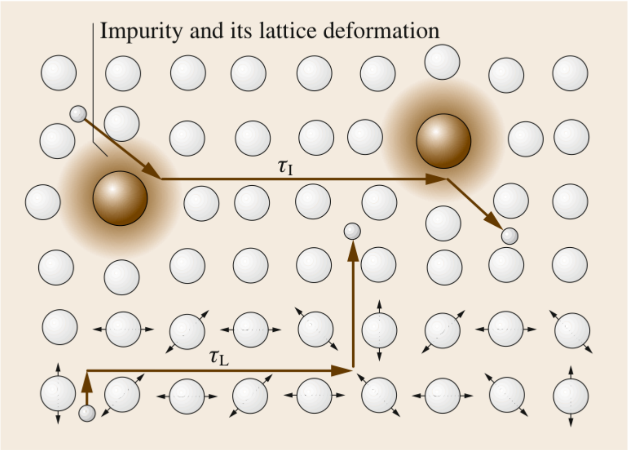


Fig. 2.3 Scattering from lattice vibrations alone with a mean scattering time τ_L, and from impurities alone with a mean scattering time τ_I

Experiment shows that the following is correct:

if different independent microscopic processes underlie the collisions of electrons at the Fermi surface the probalities of the collisios are additive

SPECTRAL DENSITY OF STATES FOR e^-

Distribution of e^- states by energy is one of the most important things to characterize the electronic spectrum excitations.

Lets introduce an important parameter THE DENSITY OF STATES.

if $N(E)$ is the # of states with energy $\leq E$ ⇒

$dN(E)$ is the # of states between E and $E+dE$

Spectral density of states $\nu(E)$ (sometime $g(E)$ or $D(E)$)

$$= \underbrace{\frac{dN(E)}{dV}}_{dn(E)} \Big/ dE = \frac{dn(E)}{dE}$$

clearly we have a connection to $\varepsilon(p)$

e.g. recal for free electrons

$$n^{3D} = \frac{2m_0E_F(0)]^{3/2}}{3\pi^2\hbar^2} \Rightarrow$$

$$D^{3D} = \frac{dn}{dE} = \frac{(2m_0)^{3/2}\sqrt{E_F(0)}}{2\pi^2\hbar^3} = \frac{\sqrt{2}\,m_0^{3/2}\sqrt{E_F(0)}}{\pi^2\hbar^3}$$

⇒ the carrier density $n = \int_0^{E_F(0)} \frac{dN}{dE}\,dE =$

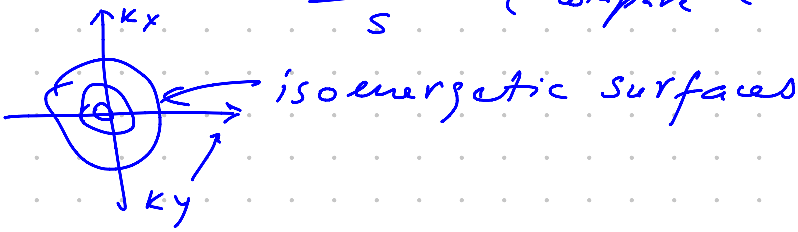
⇒ For finite T, we need to include

$$n = \int_0^\infty f\,E_n \frac{dn}{dE}\,dE$$

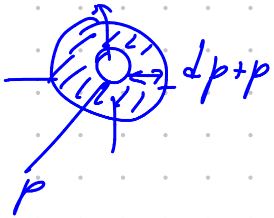
Lets derive the expression for 2D

For 1 state the area is $\frac{(2\pi\hbar)^2}{S}$ (compare $(2\pi\hbar)^3/V$)

for $E = \frac{p^2}{2m^*}$



The # of states



$$dN_p^{2D} = \frac{4\pi(p+dp)^2 - 4\pi p^2}{(2\pi\hbar)^2}$$

$$= \frac{4\pi S p\,dp}{(2\pi\hbar)^2}$$

and b/c $dE = \frac{\cancel{2}p\,dp}{\cancel{2}m^*} \Rightarrow$

$dp = \frac{m^*}{p}dE$

$p\,dp = m^* dE$

$$\Rightarrow dN_p^{2D} = \frac{4\pi S m^* dE}{(2\pi\hbar)^2} =$$

$$= \frac{4\pi m^* dE}{(2\pi\hbar)^2}$$

$$\boxed{\nu^{2D}(E) = \frac{m^*}{\pi\hbar^2}}$$

! it is independent of Energy for 2D and is defined only by the effective mass of m^* (of course for parabolic $E = p^2/2m^*$)

In general ←
however m^* depends on E
$\Rightarrow \nu^{2E}$ also depends on E

Q₁: Show for anisotropic $E = p_x^2/2m_x + p_y^2/2m_y$

$$\nu^{2D}(E) = \frac{\sqrt{m_x m_y}}{\pi\hbar^2}$$

Q₂: Show that for 1D:

$$\nu^{1D}(E) = \frac{L\sqrt{m^*}}{\sqrt{2}\,\pi\hbar\sqrt{E}}$$

L is the length of the xtal.

Van Hove singularities:

I still work on this to find an easy way to explain it (square!)

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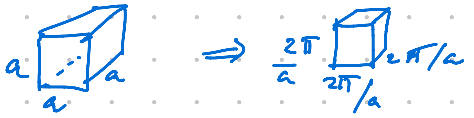
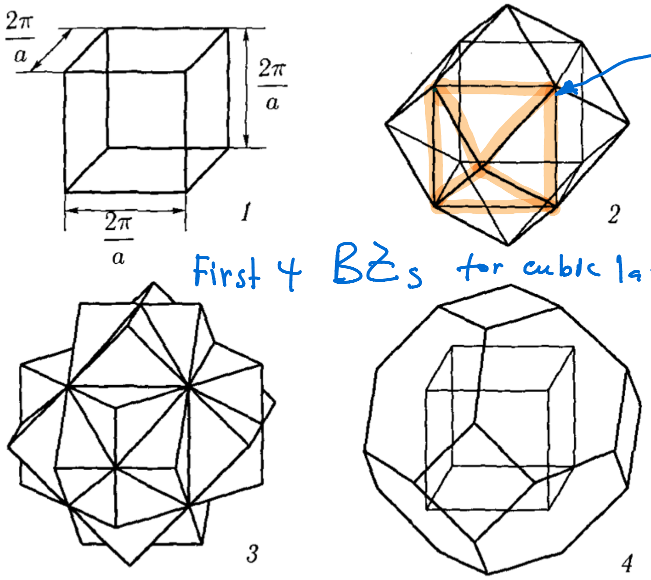
It is clear that to calculate $\nu(E)$ one needs to know $E(\bar{p})$. However we can try to get an idea what happens when we fill up the BZ.

Consider a square lattice with a period a:



. As seen 2^d BZ is made of 6 pyramides built on the faces of the 1st BZ



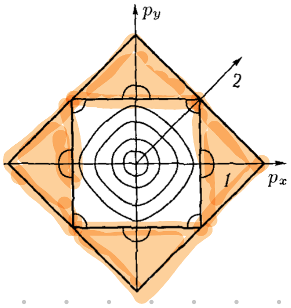
First 4 BZs for cubic lattice

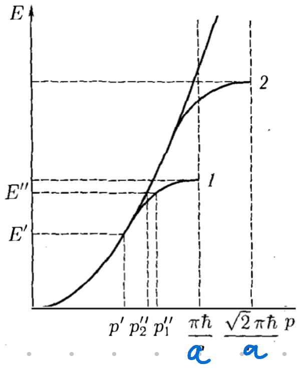
The min of energy is at $\Gamma=0$ for 1^{st} BZ

The dispersion $p^2/2m^*$ and $\nu_1^{3D}(E) \sim \sqrt{E}$

Note for $E=0$ $\frac{d\nu}{dE} \uparrow \infty$

With increasing E the sphere gets distorted when it approaches the 2D BZ





Renormalization of density of states due to electron-phonon coupling.

The polarization of lattice leads to the fact that m^*_{ij} is different from $\frac{1}{\frac{\partial^2 E}{\partial k_i \partial k_j}}$

and often can be written as $m^* = m_{band}(1+\lambda)$

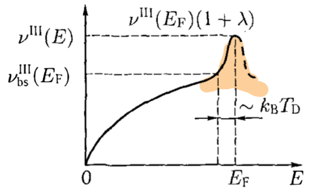
λ = electron-phonon coupling

m gets renormalized

Since $m \uparrow \Rightarrow \nu(E)$ goes up near the Fermi surface within $k_B T_D$; since $\nu^{3D}(E) \sim (m^*)^{3/2} \Rightarrow$

$$\nu^{3D}(E) = (1+\lambda)^{3/2}\, \nu_{band}(E) \Rightarrow$$

near E_F we get a peak



λ	Na	Al	Pb
	0.2	0.4	1.6

- Also e-ph coupling reduces E_F since $E_F \sim \frac{1}{m^*}$

Second we know for T=0 K $n = \int_0^{E_F} \nu^{3D}(E)\, dE$

if $\nu^{3D} \uparrow$ E_F must go down

- Also λ doesn't affect the size of FS. as it only depends on $\underline{n}$ and the lattice geometry.

- B/c of λ since $\nu^{3D}(E) \uparrow$ within $k_B T_D$ the Fermi velocity goes down by $(1+\lambda)^{3/2}$.

Proof: $|v| = \frac{\partial E}{\partial p} = \frac{\partial E}{\partial N}\frac{\partial N}{\partial p} \Rightarrow v = \frac{\partial N}{\partial p} \cdot \frac{1}{\nu(E)} \approx \frac{1}{(1+\lambda)^{3/2}}$

Electronic heat capacity and thermal Conductivity

24

The energy of quasiparticles is given

$$E = \underset{\substack{\uparrow \\ \text{two branches} \\ \text{one for electrons} \\ \text{and another for holes}}}{2} \cdot \frac{\overset{\nearrow \uparrow\downarrow \text{ spin}}{2}\, V}{(2\pi\hbar)^3} \int_{p_F}^{\infty} E(p) f_{E_n} \, 4\pi p^2 dp$$

electronic heat capacity =

$$= \frac{1}{V}\left(\frac{\partial E}{\partial T}\right)_V = 4\int_{p_F}^{\infty} E(p) \frac{\partial f_n}{\partial T} \frac{4\pi p^2 dp}{(2\pi\hbar)^3}$$

using $\nu^{3D}(E) = \frac{\sqrt{2}\, m^{*3/2}\sqrt{E}}{\pi^2\hbar^3}$, for isotropic model $\nu^{3D}(\varepsilon_F) = p_F m^* / \pi^2\hbar^3$

Q: Show that $\frac{1}{V}\left(\frac{\partial E}{\partial T}\right)_V = \int \dots = \frac{\pi^2}{3} k_B^2 \, \nu^{3D}(E_F)\, T$

and using $\nu^{3D}(E_F) \Rightarrow \boxed{C_e = \frac{p_F m^*}{3\hbar^3} k_B^2 T}$

Thus we get $C_e(T) \sim T$ by measuring C_e we can estimate $\frac{p_F m^*}{3\hbar^3}$ or m^*

In fact this is the most popular method for geting m^* in crystals.

- Note E is the sum of all excitations. Consider The averge energy of one excitation $\sim k_B T$ and their # scales with the with of $dE_F \sim k_B T$

Since $\varepsilon_F = p_F^2/2m^* \rightarrow d\varepsilon_F = p_F dp_F/m^* \rightarrow dp_F = m^* d\varepsilon_F / p_F$
$\approx m^* k_B T / p_F$

The total energy of elementary excitation inside dE is $E \sim N k_B T$

and the # of states is

see bottom of 24

$$N = \frac{4\pi p_F^2 \, dp_F}{(2\pi\hbar)^3} \approx \frac{4\pi p_F^2}{(2\pi\hbar)^3} \, \frac{m^* k_B T}{p_F}$$

and thus the energy $E \sim N k_B T = \dfrac{4\pi p_F \, m^* k_B^2 T^2}{(2\pi\hbar)^3}$

$$\Rightarrow \quad C_e = \frac{\partial E}{\partial T} \sim \frac{8\pi p_F k_B^2 m^* T}{(2\pi\hbar)^3} = \boxed{\frac{p_F \, m^* k_B^2 T}{\pi^2 \hbar^3}}$$

for $k_B T \ll E_F$

Lets compare it to the Debye expression for C_e

$$C^{3D}_{Debye} = \frac{12\pi^4 k_B N}{5}\left(\frac{T}{T_D}\right) \approx 234 \, k_B N \left(\frac{T}{T_D}\right)^3$$

$$\frac{p_F \, m^* k_B^2 T}{\pi^2 \hbar^3} \longleftrightarrow 234 \, k_B N \frac{T^3}{T_D^3} \Longrightarrow$$

$$C^{3D}_{Debye} = C_e \quad \text{at} \quad T' \sim 0.145 \sqrt{\frac{k_B T_D}{E_F}} \, T_D$$

meaning that if $T < T'$ the main contribution comes from electrons! Lets estimate when it happens

$$E_F = \frac{p_F}{2m_0} \sim \frac{1}{2m_0}\left(\frac{\pi\hbar}{a}\right)^2 \quad \text{and} \quad \theta_{Debye} \sim \frac{2\pi v_a}{a}$$

with $v_a \sim 10^5$ cm/s and $a = 10^{-8}$ cm $T'/T_D \sim 3\cdot 10^{-3}$ and $T \sim 0.5$ K. Which means that C_e dominates at $T \sim T_{L\,Helium}$.

Electronic Thermal Conductivity

TC in metals is made of γ_{e^-} and $\gamma_{phonons}$

so $\gamma = \gamma_e + \gamma_{ph}$

As for electric conductivity $\overset{\text{heat resistance}}{W_e} = \frac{1}{\gamma_e}$ and in general

$W_e = \sum_i W_{e_i}$. Notice under thermal scattering we assume such processes when after interaction the quasiparticle loses a portion of heat energy ~ k_BT but gets extra momentum due to heat gradient

Every act like this is characterized by τ_i and

$l_i = v_g \tau_i$

and like for electricity $1/\tau_e = \sum_i \left(\frac{1}{\tau_e}\right)_i$

Lets assume a gas model:

$$\gamma_e = \frac{1}{3} C_e v l \qquad l = v\tau \Rightarrow$$

$$\gamma_e = \frac{1}{3} C_e v^2 \tau$$

As shown on p.25 $C_e \sim T$ and $v \sim v_F \Rightarrow$

$$W_e = \frac{1}{\gamma_e} \sim \frac{1}{T v_F^2 \tau} = \frac{1}{T v_F^2}\left(\frac{1}{\tau_{e-ph}} + \frac{1}{\tau_{e-defect}} + \frac{1}{\tau_{e-e}}\right) = \frac{1}{T v_F^2}\left(\nu_{e-p} + \nu_{e-d} + \nu_{e-e}\right)$$

Notice for thermal conductivity charge plays no role so for the 1^{st} approx. holes and electrons are the same.

Since the average energy of quasiparticles k_BT the single scattering may lead to exchange ~ k_BT

This means that the probality of scattering on phonons is largely defined by the # of phonons.

For good crystals, the number of ph. is always >> defects and the last term is usually << $\frac{1}{\tau_{e-p}}$ and $\frac{1}{\tau_{e-b}}$

So for high enough $T \geq T_D$

$$W_e \sim \frac{\nu^N_{e-p} + \nu^U_{e-p}}{T v_F^2}$$

For $T > T_D$ the # of phonons $\sim T$ and thus

W_e is independent of T!
$T \gtrsim T_D$

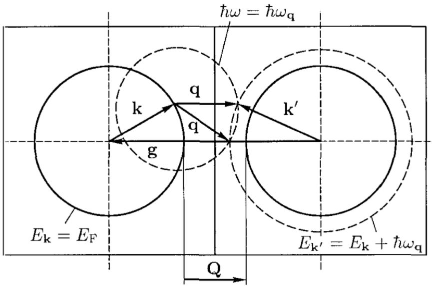
For $T \ll T_D$ $\nu^N_{e-p} \sim T^3$ and the number of scatterings is determined by the # of phonons with $q > Q$

$$\nu^U_{e-p} \sim e^{-\frac{v_a Q}{k_B T}}$$

$\Downarrow$

$$W_e \sim \frac{1}{T v_F^2}\left(\alpha T^3 + \beta e^{-\frac{v_a Q}{k_B T}}\right)$$

$$\sim \frac{1}{v_F^2}\left(\alpha T^2 + \frac{1}{T}\beta e^{-v_a Q/k_B T}\right)$$



In the area where we can neglect U processes compared to N we get

$W_e \sim T^2$ and $\gamma_e \sim T^{-2}$. For very low T $\nu_{e-def} \gg \nu_{e-p}$

and l_e, τ_e and ν_e become T-independent.

Thus we get in this area $\gamma_e \sim T$ → (all terms inside the bracket T-independent)