Problems

1.1 To find the behavior of the heat capacity at the second order transition, recall the definition of Φ_{min} as $\Phi_{min} = \phi_o - \frac{a^2}{4B}(T - T_c)^2$. In this case, we'll first consider the range of temperatures in which $T < T_c$. Given that $S = -\frac{\partial\Phi}{\partial T}$, we can calculate the entropy as $S = S_o + \frac{a^2}{2B}(T - T_c)$. Now that we know the entropy, we can calculate the specific heat, since $c_p = T(\frac{\partial S}{\partial T})$. Doing this, we get that $c_p = T(\frac{a^2}{2B})$.

Now, let's consider the case in which $T > T_c$. In this circumstance, $\Phi_{min} = \phi_o$, so the entropy $S = S_o$. If the entropy is constant, then the specific heat in this range is simply equal to zero, by the above relation. Thus, we can conclude that the change in specific heat at the second order transition is

$$\Delta c_p|_{T=T_c} = \frac{a^2 T_c}{2B} - 0 = \frac{a^2 T_c}{2B}$$

1.2 To find the entropy in the ordered state, let's take our results from 1.1, and substitute it into our equation. Specifically, ordered state entropy is defined as $S_{ord} = \int_{T=0}^{T_c} \frac{c_p(T)}{T} dT$. This is relatively straightforward, and what we get is that $S_{ord} = \frac{a^2 T}{2B}|_0^{T_c} = \frac{a^2 T_c}{2B}$. Now, for a spin S system, the total entropy in the disordered system is given by $S = k_b \ln(2S + 1)$, where k_b is Boltzmann's constant. Thus, the total disordered entropy for this spin 1/2 system is given as $S = k_b \ln(2)$.

1.3 From a physical standpoint, if one were to measure the total entropy of a spin 1/2 system and find that it is lower than the calculated S_{ord}, it would mean that certain parts of the system are ordered, even above the critical temperature. This might refer to orderings that occur at short distances across the measured system. On the other hand, if the total entropy is larger than the calculated S_{ord}, it means that there are other degrees of freedom that are affected with temperature, and that most likely order themselves at T_c. In this case, it means that the model does not fully encompass the orderings of the system.

2 To calculate the relationship between the thermal expansion coefficient β and Δc_p, recall that $\Delta c_p = \frac{a^2 T_c}{2B}$ and that $S = -\frac{\partial \Phi}{\partial T}$. By definition, second order phase transitions are continuous as a function of entropy. Thus, we can write a differential relation $\Delta(\frac{\partial S}{\partial P}) + \frac{\partial T}{\partial P}\Delta(\frac{\partial S}{\partial T}) = 0$, or written another way, $\Delta(\frac{\partial S}{\partial P}) = -\frac{\partial T}{\partial P}\Delta(\frac{\partial S}{\partial T})$. To write this in terms of the quantities we want, let's take the derivative of S with respect to P, giving us $\frac{\partial S}{\partial P} = -\frac{\partial^2 \Phi}{\partial P \partial T}$. We can rewrite this as $\frac{\partial S}{\partial P} = -\frac{\partial}{\partial T}\frac{\partial \Phi}{\partial P}$. Since $\frac{\partial \Phi}{\partial P} = V$, we can say that $\frac{\partial S}{\partial P} = -\frac{\partial V}{\partial T}$. Furthermore, since $c_p = T(\frac{\partial S}{\partial T})$, we can substitute this into our first differential relation to get that $\Delta \frac{\partial V}{\partial T} = \frac{\partial T}{\partial P}\Delta \frac{c_p}{T}$. However, $\frac{\partial V}{\partial T} = \beta V$, so we finally get that

$$\Delta \beta = \frac{\partial T}{\partial P}\frac{\Delta c_p}{VT}$$

To calculate the relationship between the compressibility κ and Δc_p, we'll use a very similar method to the above. Since volume also does not change along a second order phase transition, we can write the differential relation $\Delta \frac{\partial V}{\partial P} = -\frac{\partial T}{\partial P}\Delta \frac{\partial V}{\partial T}$. By definition, compressibility is equal to $\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}$. This, combined with the definition of β given above, allows us to rewrite the differential relation to read $\Delta \kappa = -\frac{\partial T}{\partial P}\Delta \beta$. Finally, we can use our result for $\Delta \beta$ to write this as

$$\Delta \kappa = -(\frac{\partial T}{\partial P})^2 \frac{\Delta c_p}{VT}$$