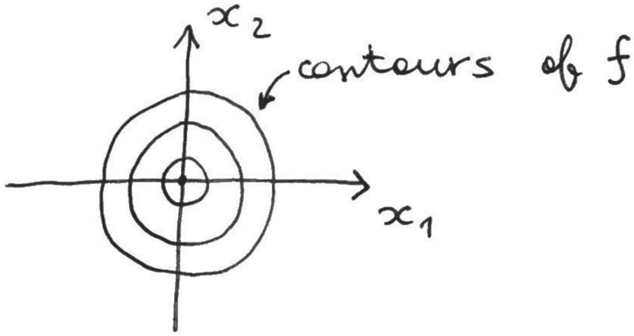
[Lagrange multipliers : a representative example]

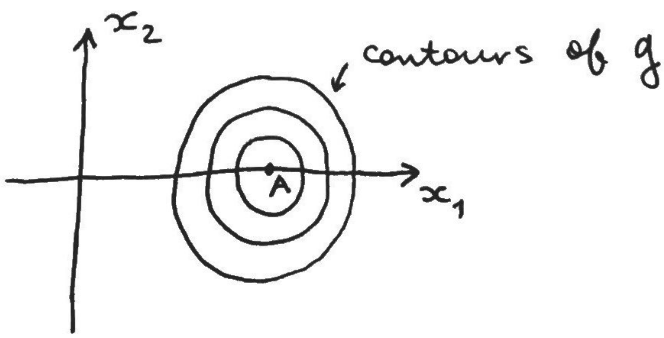
Consider $f(x_1, x_2) = 1 - x_1^2 - x_2^2$



Clearly, $f(x_1, x_2)$ is maximized at $(x_1^*, x_2^*) = (0, 0)$, with $f(x_1^*, x_2^*) = 1$ in the absence of any constraints.

Now let's impose an inequality constraint : $g(x_1, x_2) \geq 0$, where

$g(x_1, x_2) = 1 - x_2^2 - (x_1 - A)^2$ $A \geq 0$ for concreteness



There're 2 cases to consider:

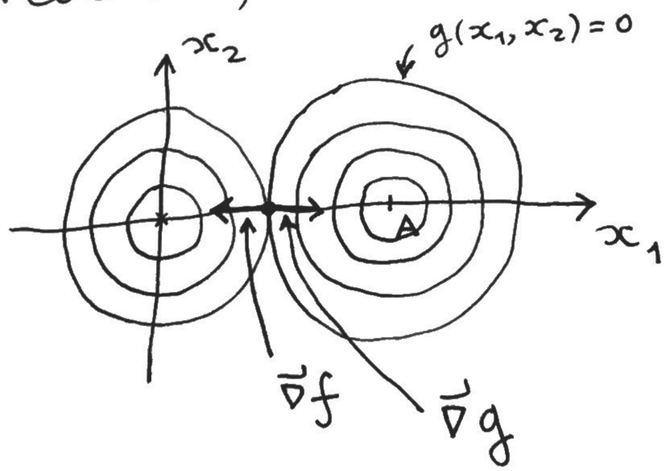
1. $g(0,0) = 1 - A^2 \geq 0 \Rightarrow A \leq 1$

Then the constraint is automatically satisfied (i.e. it is inactive), $\lambda = 0$, and $(0,0)$ is still the solution.

2. $A > 1$ (i.e., $g(0,0) < 0$)

Then the constraint is active and the max of $f(x_1, x_2)$ will be on the boundary: $g(x_1, x_2) = 0$ & $\lambda \neq 0$.

Moreover, $\lambda > 0$:



$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$$

Note that $\vec{\nabla} f$ is $\perp$ $g(x_1, x_2) = 0$ surface [otherwise we could have increased the value of f by moving along the $g(x_1, x_2) = 0$ surface, w/out breaking the constraint].

Likewise, $\vec{\nabla} g$ is $\perp$ $g(x_1, x_2) = 0$ surface:

$$\underbrace{g(\vec{x} + \vec{\xi}) \simeq g(\vec{x})}_{\text{both } \vec{x} \text{ \& } \vec{x}+\vec{\xi} \text{ are on the surface } g(x_1,x_2)=0} + \vec{\xi}^T \cdot \vec{\nabla} g(\vec{x})$$

implies $\vec{\nabla} g = 0$ in the $\vec{\xi} \to 0$ limit.

Define $\mathcal{L} = f(x_1, x_2) + \lambda g(x_1, x_2)$,

then $\vec{\nabla} f + \lambda \vec{\nabla} g = 0$ implies $\lambda > 0$

in this case, since $\vec{\nabla} f \uparrow\downarrow \vec{\nabla} g$

It's clear that for this system (x_1^*, x_2^*) will be somewhere on the x_1 axis (i.e., $x_2^* = 0$).

Let's see how to get this through $\mathcal{L}$:

$$\mathcal{L} = (1 - x_1^2 - x_2^2) + \lambda(1 - x_2^2 - (x_1 - A)^2)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = 0 \Rightarrow x_1 + \lambda(x_1 - A) = 0, & (1) \\ \dfrac{\partial \mathcal{L}}{\partial x_2} = 0 \Rightarrow x_2 + \lambda x_2 = 0, & (2) \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow 1 - x_2^2 - (x_1 - A)^2 = 0. & (3) \end{cases}$$

Eq. (2) yields $x_2^* = 0$ as expected, since $\lambda > 0$.

Eq. (1) yields $x_1^* = \dfrac{\lambda A}{1+\lambda}$.

Finally, Eq. (3) gives

$$\left(\frac{\lambda A}{1+\lambda} - A\right)^2 = 1 \text{, or}$$

$$\left(\frac{A}{1+\lambda}\right)^2 = 1 \implies \lambda = A - 1 \underbrace{> 0}_{\text{positive}}$$

We have $x_1^* = \dfrac{(A-1)A}{A} = \underline{\underline{A-1}}$, and

$$(x_1^*, x_2^*) = \left(A-1, 0\right) \quad \text{as expected from symmetry}$$

Indeed, $g(x_1^*, x_2^*) = 0$ while

$f(x_1^*, x_2^*) = 1 - (A-1)^2$, worse than the unconstrained max $f(0,0) = 1$.

Finally, cases 1 & 2 together imply that

$$\begin{cases} \lambda \geq 0, \\ g(\vec{x}) \geq 0, & \text{KKT conditions} \\ \underbrace{\lambda g(\vec{x}) = 0} \end{cases}$$

either $\lambda = 0$ (constraint inactive)

or $g(\vec{x}) = 0$ (constraint active)